Types in o-minimal theories

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by

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Abstract

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We extend previous work on classifying o-minimal types, and develop several applications. Marker developed a dichotomy of o-minimal types into "cuts" and "noncuts," with a further dichotomy of cuts being either "uniquely" or "non-uniquely realizable." We use this classification to extend work by van den Dries and Miller on bounding growth rates of definable functions in Chapter 3, and work by Marker on constructing certain "small" extensions in Chapter 4.

We further sub-classify "non-uniquely realizable cuts" into three categories in Chapter 2, and we give define the notion of a "decreasing" type in Chapter 5, which is a presentation of a type well-suited for our work. Using this definition, we achieve two results: in Chapter 5.2, we improve a characterization of definable types in o-minimal theories given by Marker and Steinhorn, and in Chapter 6 we answer a question of Speissegger's about extending a continuous function to the boundary of its domain. As well, in Chapter 5.3, we show how every elementary extension can be presented as decreasing.

Chapter 1

Background

1.1 Introduction

This paper presents several new results on types in o-minimal structures – ordered structures in which every definable subset is a finite union of intervals and points. Because o-minimal structures are not simple (in the terminology of [She80]), the study of o-minimal types cannot directly avail of many of the techniques developed for types in stable theories. Work was done on developing analogous techniques – see [Ons06] for a rank that works for o-minimal structures, or [Dol04], for a complete description of forking in o-minimal theories. Another promising avenue was that of definable types.

Definition 1.1.1. A type, $p \in S(M)$, is *A-definable*, for $A \subseteq M$, if, for any formula $\varphi(x, b)$, with $b \in M$, there is an A-definable formula $d_\varphi(y)$, such that $\varphi(x, b) \in p \iff d_\varphi(b)$.

The definable 1-types in o-minimal structures were explicitly named in [Mar86], and a complete description of definable n-types given in [MS94]. The methods of categorization used in [Mar86] can be used to yield several new results. In Chapter 3, we extend a result of [vdDM96], showing that, for any o-minimal structure, if a function is definable in an elementary extension, that function is bounded by a function definable in the original structure. In Chapter 4, we give an example of a pair of o-minimal structures in which the larger realizes no new "finite" types over the smaller one.

The categorization of [Mar86] can be extended, yielding finer results. In the latter part of the paper, we define the notion of a *decreasing type* – one in which each element is either infinitesimal over all the elements before it, or at the same scale. Using the predictability that such types give us, we are able to improve the description of "definable n-type" given in [MS94], as well as prove a number of results about the existence of decreasing types. We also explore the strong connection between decreasing types and valuations on o-minimal fields.

Finally, in Chapter 6 we apply decreasing types to answer a question of Patrick Speissegger's, giving complete conditions for continuously extending a definable function to a type "near" a boundary point of the function's domain. A corollary to this result gives conditions for extending a definable function continuously onto a (non-definable) curve whose limit point is not in the function's domain.

1.2 Preliminaries and Notation

We will always let T denote the complete theory we are currently discussing. We adopt the usual convention of, given T, fixing a "monster model," $\bar{M}$, saturated of a sufficient cardinality that all sets and structures we consider can be assumed to be subsets and elementary substructures of $\bar{M}$.

If A is a subset of a topological space, let $\bar{A}$ be the geometric closure of A. Let I be an ordered set (ordered by $<$), and let $f : I \to \bar{M}$ be an injective function. Then we let $\langle f(i)\rangle_{i\in I}$ denote the sequence indexed by I with $f(i$ appearing before $f(j)$ iff $i < j$. Similarly, if a and b are sequences, $\langle a, b\rangle$ denotes the sequence that is the concatenation of a with b, and likewise if a_j, $j \in J$, are sequences, with J some ordered set, then $\langle a_j\rangle_{j\in J}$ is the concatenation of all these sequences.

If $c = \langle c_i\rangle_{i\in I}$ is a sequence, with ordered index set I, then, for $i \in I$, $c_{<i}$ denotes $\langle c_i\rangle_{i\in J}$, where $J = \{j \mid j < i\}$, with the induced order from I. Similarly for $c_{>i}$, $c_{\leq i}$, etc. If $C \subseteq \bar{M}^n$ is a set, we define

$$\pi_{\leq i}(C) := \{x \in \bar{M}^i \mid \exists y \in \bar{M}^{n-i}(\langle x, y\rangle \in C)\}.$$

Likewise, we define

$$\pi_{>i}(C) := \{\exists y \in \bar{M}^{n-i} \mid x \in \bar{M}^i(\langle x, y\rangle \in C)\}.$$

As well, for $a \in \bar{M}^k$ $(k < n)$, we define

$$C_a := \{y \in M^{n-k} \mid \langle a, y\rangle \in C\}.$$

If f is an $m + n$-ary function, then, if $c \in \pi_{\leq m}(\mathrm{dom}(f))$, f_c is the n-ary function, $f(c, -)$. As a convention, all variables will be assumed to be tuples, unless otherwise stated, or clear from context. Subscripted variables (when referring to the ith element of a sequence) are always singletons.

1.3 O-minimal theories

Definition 1.3.1. Let M be a structure in a language, L, containing a symbol $<$ that is interpreted in M as a transitive, irreflexive, antisymmetric binary relation − an order. The structure M is *o-minimal* if, for any formula $\varphi(x, a)$, with $a \in M$ a tuple, the set $\{b \in M \mid M \models \varphi(b, a)\}$ is equal to a finite union of points and intervals (with endpoints in $M \cup \{\pm\infty\}$).

Definition 1.3.2. A complete theory, T, is o-minimal if it has an o-minimal model.

All results in this chapter are for a theory, T, that is o-minimal, expanding the theory of a dense linear order without endpoints. We give the results that we will use in this work, and refer the reader to [vdD98] for most proofs, and a complete background on o-minimal structures.

A fundamental result in o-minimal structures is that of "cell decomposition." First, we define a cell.

Definition 1.3.3. (Based on Chapter 3, 2.2 of [vdD98]) A 0-*cell* is a point. Given an n-cell, C, an $n+1$-*cell* has one of two forms:

1. $\{\langle x, r\rangle \in C \times \bar{M} \mid f(x) < r < g(x)\}$, or
2. $\{\langle x, f(x)\rangle \in C \times \bar{M}\}$,

where f and g are definable (over some parameters) n-ary functions whose domains include C. A *cell* is an n-cell for some n. We say that a cell is A-definable if all the functions (and initial point) used to define it are A-definable.

Definition 1.3.4. A cell, C, is *regular* if, whenever $x, y \in C$, and x and y differ only on the ith coordinate, then the line connecting x and y is entirely contained in C. It is a version of convexity, but only coordinate-by-coordinate.

Theorem 1.3.5. *(Chapter 3, 2.19, Exercises 2,4 of [vdD98])*

1. *Given any definable sets $A_1, \ldots, A_k \subseteq \bar{M}^m$, for any m, there is a partition of $\bar{M}^m$ into cells that partitions each of $A_1, \ldots, A_k$. Moreover, these cells are definable over the parameters used to define $A_1, \ldots, A_k$.*
2. *For each definable function $f : A \to \bar{M}$, $A \subseteq \bar{M}^m$, there is a partition of A into regular cells such that, for each cell B, the restriction $f \upharpoonright B : B \to \bar{M}$ is continuous and monotonic in each coordinate.*

Proof. (Sketch) The standard proof of cell decomposition gives the first claim, and the second without the assumptions that the cells are regular or that f is monotonic on each coordinate. However, it is not hard to see that the cells can be made regular, by further subdividing so that each boundary function is monotonic on the projection of each cell, by induction. Then they can be futher subdivided to make f monotonic, since, by o-minimality, f can change its coordinate-by-coordinate behavior only finitely many times on a cell. $\square$

This theorem is integral to all we do going forward. We will habitually FIXME (written note says to explain why?) just say “taking a cell decomposition” to refer to applying this theorem.

Theorem 1.3.6. *The pure theory of real closed fields in the language* $(+, \cdot, 0, 1, <)$ *has quantifier elimination.*

Proof. This is [Hod93], Theorem 8.4.4. $\square$

Definition 1.3.7. Let $\mathrm{dcl}(A)$ denote the definable closure of A – the set of elements that satisfy $\varphi(x)$, for some A-definable formula A, such that $\models \exists! x \varphi(x)$. See [Hod93], 4.1, for more details.

Lemma 1.3.8. $\mathrm{dcl}(A) = \mathrm{acl}(A)$.

Proof. Inclusion in one direction is trivial, so it remains to show that, if $b \in \mathrm{acl}(A)$, then $b \in \mathrm{dcl}(A)$. Let b satisfy $\varphi(x)$, and let $b_1 < \ldots < b_n$ be the only elements satisfying $\mathrm{acl}(A)$. Let $b_i = b$. Then the formula $\varphi(x) \wedge \exists^{=i-1} y(\varphi(y) \wedge y < x)$ is satisfied only by b. $\square$

Lemma 1.3.9. *Let A be a set, and let $c = \langle c_1, \ldots, c_n \rangle$. Let $d_1, \ldots, d_m \in \mathrm{dcl}(Ac)$. Then $d_1, \ldots, d_m$ has at most n algebraically independent elements over A.*

Proof. We assume that $c_1, \ldots, c_n$ are algebraically independent over A – if not, discard non-independent elements. We also assume that $d_1, \ldots, d_m$ are algebraically independent over A. For each d_i, there is an f_i, a k_i-ary function, with k_i minimal, such that $f_i(c_{j(i)_1}, \ldots, c_{j(i)_{k_i}}) = d_i$, for $j(i)_1 < \ldots j(i)_{k_i} \leq n$. We proceed to exchange c_i's for d_i's in stages, constructing a tuple e^i at each stage. Set $e^0 = c$. At stage i, we may reorder $e^{i-1}_{\geq i}$ so that e^{i-1}_i is the c_j with the smallest j still remaining such that, for some $l < k_i$, $j = j(i)_l$. Thus, at stage 1, we reorder e^0 so that $e^0_1 = c_{j(1)_1}$. By exchange, $\mathrm{dcl}(Ae^0) = \mathrm{dcl}(Ad_1 e^0_{>i})$. Let $e^1 = \langle d_1, e^0_{>1} \rangle$. Similarly, at stage i, we know that e^{i-1}_i is the first remaining c_j such that f_i depends on c_j. Such a c_j must exist, else $d_i \in \mathrm{dcl}(Ad_{<i})$, contradiction. Then exchange d_i for c_j. After n steps, this process yields $d_1, \ldots, d_n$, independent, with $\mathrm{dcl}(Ad_1 \ldots d_n) = \mathrm{dcl}(Ac)$. But then $m \leq n$. □

Lemma 1.3.10. *([PS86], Theorem 3.3 (forward direction)) Let $A = \mathrm{dcl}(A)$ be a set, and let $p \in S_1(A)$. Then the formulas in p of the form $x > a$, $x < a$, and $x = a$ generate p.*

Proof. Let $\varphi(x)$ be any formula. By cell decomposition, there are elements $a_1, \ldots, a_k \in A$ and intervals $I_1, \ldots, I_m$ (with A-definable endpoints), such that

$$\varphi(x) \iff \left(\bigvee_{i \leq k} x = a_i \vee \bigvee_{i \leq m} i \in I_i \right).$$

Since p is a complete type, either for some $i \leq k$, $x = a_i$ is in p, or for some $i \leq m$, $x \in I_i$ is in p, or $\bigvee_{i \leq k} x = a_i \vee \bigvee_{i \leq m} i \in I_i$ is in p. The first two possibilities imply $\varphi(x)$ is in p, while the third implies $\varphi(x)$ is not in p. Thus, $\varphi(x)$'s membership in p is determined by the order and equality formulas in p. □

Theorem 1.3.11. *(Theorem 5.1 of [PS86]) For any set A, there is a structure, M ,with $A \subseteq M$, and such that, for any structure N with $A \subseteq N$, M elementarily embeds into N. The structure M is unique up to isomorphism, and so we denote it* $\mathrm{Pr}(A)$*. If M is a structure, and A is a set, we may denote* $\mathrm{Pr}(MA)$ *by $M(A)$.*

Proof. (Sketch) In most cases, $\mathrm{Pr}(A)$ will be $\mathrm{dcl}(A)$. The reason is that T will usually have Skolem functions. We show this by considering any sentence $\exists x \varphi(x, a)$, where a is a tuple. The set satisfying $\varphi(x, a)$ is given by a finite union of points and intervals. We may definably choose an point satisfying $\varphi(x, a)$, if such point exists in the decomposition, and even do so uniformly in a. If no such isolated points exist, we must choose a point in the interior of an interval. In the case where T expands the theory of an ordered group, we may do that using the average of the endpoints, or a similar means. Thus, we will have Skolem functions, and hence a prime model via Theorem 3.1.1 of [Hod93], which gives the Skolem hull – an elementary substructure of $\bar{M}$ containing A that must be contained in any other elementary substructure containing A – hence, a prime model. If there is no definable way to choose a point in the interior of an interval, then an arbitrary choice for each such homogeneous interval will yield the prime model. □

Lemma 1.3.12. *Let f be an A-definable function, defined on a neighborhood above a, (a,b), for some $b \in \operatorname{dcl}(A) \cup \{\infty\}$, with $a \in \operatorname{dcl}(A) \cup \{-\infty\}$. If f is bounded on (a,b), then $\lim_{x\to a^+} f(x) \in \operatorname{dcl}(A)$. Similarly if f is defined on a neighborhood below a (with $a \in \operatorname{dcl}(A) \cup \{\infty\}$).*

Proof. The formula

$$\varphi(y) := \forall c, d\, (c < y < d \Rightarrow \exists z \forall x \in (a, z)\, (f(x) \in (c, d)))$$

shows that, if the limit exists, it is in $\operatorname{dcl}(A)$, since φ holds on the limit, and φ is A-definable. By [vdD98], Chapter 3, 1.6 (Corollary 1), $\lim_{x\to a^+} f(x)$ exists, though it is possibly infinite. However, since f is bounded, the limit cannot be infinite, and so we are done. $\square$

Lemma 1.3.13. *Let $S' \subseteq S$ be definable sets in $\bar{M}^{m+n}$, and let $A \subseteq \bar{M}^m$ be definable such that S'_a is open (closed) in S_a for all $a \in A$. Then there is a partition of A into definable subsets $A_1, \ldots, A_k$ such that $S' \cap (A_i \times \bar{M}^n)$ is open (closed) in $S \cap (A_i \times \bar{M}^n)$, for $i = 1, \ldots, k$.*

Proof. This is [vdD98], Chapter 6, Corollary 2.3. $\square$

Lemma 1.3.14. *Let S' be a definable set in $\bar{M}^{m+n}$. Let $S = \{x \mid \exists a \in \pi_{\leq m}(S)(x \in \overline{\{a\} \times S_a})\}$. Then there is a partition of $\bar{M}^m$ into definable subsets $A_1, \ldots, A_k$ such that $\overline{S'} \cap (A_i \times \bar{M}^n) = S \cap (A_i \times \bar{M}^n)$, for $i = 1, \ldots, k$. In other words, the fiber of the closure is the closure of the fiber.*

Proof. S and $\overline{S'}$ satisfy the conditions of Lemma 1.3.13, with $A = \bar{M}^m$, so we can find $A_1, \ldots, A_k$ such that $S \cap (A_i \times \bar{M}^n)$ is closed in $\overline{S'} \cap (A_i \times \bar{M}^n)$, which implies that the two sets are equal, for each $i = 1, \ldots, k$ $\square$

Lemma 1.3.15. *Let $S \subseteq \bar{M}^{m+n}$ be definable, $f : S \to \bar{M}^k$ a locally bounded definable map, and $A \subseteq R^m$ a definable set such that for all $a \in A$ the map $f_a : S_a \to R^k$ is continuous. Then there is a partition of A into definable subsets $A_1, \ldots, A_M$ such that each restriction*

$$f \restriction S \cap (A_1 \times R^n) : S \cap (A_i \times R^n) \to R^k$$

is continuous.

Proof. This is [vdD98], Chapter 6, Corollary 2.4. $\square$

Lemma 1.3.16. *Let $\bar{M}$ expand an ordered group. If $a \in \overline{X} \setminus X$, where X is definable, then there is a definable continuous injective map $\gamma : (0, s) \to X$, for some $s > 0$, such that $\lim_{t\to 0} \gamma(t) = a$.*

Proof. This is [vdD98], Chapter 6, 1.5. $\square$

Lemma 1.3.17. *Let $\bar{M}$ expand an ordered group. If $C \subseteq \bar{M}^n$ is a definable bounded cell, then $\overline{\pi_{\leq n-1}(C)} = \pi_{\leq n-1}(\overline{C})$.*

Proof. This is [vdD98], Chapter 6, 1.7. $\square$

Another fundamental result in o-minimal structures is the "trichotomy theorem." While we will not use the full result, part of it will be useful.

Definition 1.3.18. An element a is *non-trivial* if there is a definable open interval I containing a and a definable continuous function $F := I \times I \to M$ such that F is strictly monotone in each variable.

Theorem 1.3.19. *([PS98], Theorem 1.1) Let M be ω_1-saturated. Let $a \in M$ be non-trivial. Then there is a convex group $G \subseteq M$ with the graph of multiplication in G given by the intersection of a definable set with G^3.*

Definition 1.3.20. Let A be any set. A *group chunk* on A is given by a binary function, $*$, with domain a subset of A^2, such that the following hold.

1. For $a, b, c \in A$, $a * (b * c) = (a * b) * c$ whenever $\langle a, b * c \rangle, \langle a * b, c \rangle \in \mathrm{dom}(*)$.
2. There is a unique element, $e \in A$, such that if $a \in \pi_1(\mathrm{dom}(A))$, then $a * e = e * a = a$ (e is the "identity element.")
3. If $a \in \pi_1(\mathrm{dom}(A))$, then there is some a' such that $a * a' = a' * a = e$.

Note: this definition is independent of o-minimality, our monster model $\bar{M}$, etc.

Corollary 1.3.21. *Let M be a structure, and let $a \in M$ be non-trivial. Then there is an M-definable binary function, $*$, such that, on some M-definable interval I, about a, $*$ defines a group chunk, with an identity element in M.*

Proof. We know that, in $\bar{M}$, there is a convex group, G, containing a, with the graph of multiplication in G, denoted $*$, given by an $L(M)$-formula, $\varphi(x, y, z, c)$, where $c \in \bar{M}$ is a tuple. We may assume that $\varphi(x, y, z, c)$ defines a function on $\bar{M}$, since, for any $b, d \in G$, there must be an isolated point, $g \in G$, such that $\varphi(b, d, g, c)$ holds, so we may assume that it is the only such point, and then that $\varphi(x, y, -, c)$ holds for a single point for any $x, y \in \bar{M}$. Now, note that the properties of a group chunk are first-order, assuming that the function giving the group chunk is definable. Thus, since the properties of a group chunk certainly hold on G, with $*$ the function, the properties of a group chunk must hold on I, for I some $\bar{M}$-definable interval containing a. Then, the sentence

$$\exists w, u_1, u_2(\varphi(x, y, z, w) \text{ defines a group chunk on } (u_1, u_2) \wedge a \in (u_1, u_2))$$

holds in $\bar{M}$, and hence in M. Since the identity element is definable from the group chunk function, we are done. $\square$

Chapter 2

Classifying O-minimal Types

2.1 Types in Ordered Structures

Cuts and Noncuts

In [Mar86], Dave Marker gave a fundamental classification of o-minimal types. It is predicated on the following dichotomy of types in densely ordered structures.

Definition 2.1.1. Let M be any densely ordered structure. If $A \subseteq M$ and $c \in M$, $\mathrm{tp}(c/A)$ is a *cut* iff there are $a, b \in \mathrm{dcl}(A)$ such that $a < c < b$, and for $a \in \mathrm{dcl}(A)$ with $a < c$, there is $a' \in \mathrm{dcl}(A)$ with $a < a' < c$, and likewise for $a > c$. Say that $\mathrm{tp}(c/A)$ is a *noncut* iff it is not algebraic and not a cut. Abusing terminology, we will also refer to c itself as a cut/noncut over A.

Note that our definition of "cut" is closely related to the traditional definition of a (Dedekind) cut. In our terminology, a Dedekind cut (A, B) will be a cut if B has no least element. However, the more ambiguous notion of a "cut" as being any type in the order is not compatible with our terminology.

While the definition of "noncut" is negative, we can actually give positive conditions:

Lemma 2.1.2. *Let $p \in S(A)$ be a noncut, with $A = \mathrm{dcl}(A)$. Then one of the following is true:*

1. *$p \models x > a$, for all $a \in A$ – p is called the noncut* near ∞, *or ∞^-;*
2. *$p \models x < a$, for all $a \in A$ – p is called the noncut* near $-\infty$, *or $-\infty^+$;*
3. *$p \models x > b$, $p \models x < a$, for all $a > b \in A$ – p is called the noncut* below a, *or a^-; or*
4. *$p \models x < b$, $p \models x > a$, for all $a < b \in A$ – p is called the noncut* above a, *or a^+.*

Proof. Clear – examine the ways that p can fail to be a cut. $\square$

We may refer to the last two kinds of noncuts as noncuts "near" a. By Lemma 1.3.10, the above formulas generate complete types, so over any set of parameters, a^+, etc., is well-defined.

Lemma 2.1.3. *If $p \in S(A)$ is a noncut near $a \in \mathrm{dcl}(A)$, then p is a-definable.*

Proof. Let $\varphi(x, y)$ be any formula, with y a tuple. WLOG, assume p is a noncut above a. The formula

$$\psi(y) = \exists d > a(\forall x \in (a, d)(\varphi(x, y)))$$

holds on b if and only if $\varphi(x, b)$ is in p, and $\psi(y)$ is a-definable. $\square$

Lemma 2.1.4. *If $p \in S(A)$ is a cut, then p is not A-definable.*

Proof. We show that if p is A-definable, then p is not a cut. We have the formula $x < a$. Since p is definable, there is some A-definable ψ such that $\psi(y)$ holds iff $x < y$ is in p. But then ψ holds on some initial segment of $\mathrm{Pr}(A)$. Let $b \in \mathrm{Pr}(A) \cup \{\infty\}$ be the right endpoint of this interval. If $b = \infty$, then p is the noncut near ∞. If b is not ∞, then there is no $b' < b$ such that $x < b'$ is in p, and there can be no $b' > b$ such that $x > b'$ is in p. Thus, p is a noncut near b, or p is the isolated type that says $x = b$. $\square$

2.2 Properties of Cuts and Noncuts

Henceforth, we restrict to o-minimal structures, and assume that T, our ambient theory, is o-minimal, expanding the theory of a dense linear order. Note that we may have further varying assumptions on T, which we will state.

Lemma 2.2.1. *Let c realize the type of a cut over A, and d the type of a noncut over A. Then there is no A-definable function, f, such that $f(c) = d$ (and thus no A-definable function such that $f(d) = c$).*

Proof. This is Lemma 2.1 and 2.2 of [Mar86]. $\square$

Lemma 2.2.2. *Let b be a noncut near α over A. Let f be A-definable such that $f(b)$ is a noncut near β over A. Then f is increasing if b and $f(b)$ are noncuts both above or both below, and f decreasing otherwise.*

Proof. We do the cases where b is a noncut above α – the cases for "below" are analogous. We know f is non-constant in a neighborhood of b, else $f(b)$ will not be a noncut over A. Suppose $f(b)$ is a noncut above β. If f is decreasing, then $f(\alpha) > f(b)$. But now there is no element of $\mathrm{dcl}(A)$ between $f(\alpha)$ and $f(b)$, or between $f(b)$ and β, and $f(b) \notin \mathrm{dcl}(A)$. This contradicts the fact that $\mathrm{dcl}(A)$ is a dense linear order. The argument if $f(b)$ is a noncut below β is similar. $\square$

Lemma 2.2.3. *(From Lemma 2.2 of [Mar86]) If $M \prec N$, with N realizing only cuts over M, and $\mathrm{tp}(c/N)$ is a cut, then $N(c)$ realizes only cuts over M.*

Proof. Suppose $N(c)$ realizes a noncut over M. We show that either $\mathrm{tp}(c/N)$ is a noncut, or N realizes a noncut over M. Let $f(c)$ be a noncut near $\alpha \in M$ over M, with f an N-definable function. If c is a noncut over N, then $f(c)$ is not a noncut over N, so there is some $d \in N$ with d between α and $f(c)$. But then d is a noncut near α over M, so N realizes a noncut over M. $\square$

Lemma 2.2.4. *If T expands the theory of an ordered field, then all noncuts over a fixed parameter set are interdefinable.*

Proof. See the Example following Definition 2.1 of [Dol04]. $\square$

Lemma 2.2.5. *Let A be a set. If, for any elements a, b, the noncut above a (over Aab) is interdefinable with the noncut above b, then all elements are non-trivial.*

Proof. By interdefinability, there is an A-definable function f, with $f(x, b, a)$ mapping an interval above a to an interval above b. It is clear that $f(x, b, a)$ must be increasing. If we let b vary, then it is also clear that $f(c, -, a)$ must be increasing, for some c sufficiently close to a. Then $f(-, -, a)$ witnesses the non-triviality of a. $\square$

Definition 2.2.6. If A has the property of Lemma 2.2.5, then we say that *parallel noncuts are interdefinable over A*. Note that, if T_A expands the theory of an ordered group, then parallel noncuts are interdefinable over A.

Lemma 2.2.7. *Let A be a set. Let b, c be any elements. If all the noncuts above and below b and c, and near $\pm\infty$, are interdefinable (over Abc), then, for any $B \supseteq A$, $\mathrm{dcl}(B)$ is dense without endpoints.*

Proof. To show that $\mathrm{dcl}(B)$ is dense without endpoints, it suffices to show that $\mathrm{dcl}(B)$ is nonempty, and, given a point, b, there are points $b^-, b^+ \in \mathrm{dcl}(Ab)$ with $b^- < b < b^+$, and, given $b < c$, there is $d \in (b, c) \cap \mathrm{dcl}(Abc)$. The argument for all three is the same – namely, we take an interval, and show that map between the noncut above the left-hand endpoint and the noncut below the right-hand endpoint yields a point in the interval definable from A and the endpoints. We apply this to the interval $(-\infty, \infty)$ to show $\mathrm{dcl}(B)$ is nonempty, to the intervals (b, ∞) and $(-\infty, b)$ to get b^+ and b^-, respectively, and to (b, c) to get d. So let (α, β) be an interval, with $\alpha, \beta \in \bar{M} \cup \{\pm\infty\}$. By hypothesis, there is an A-definable function, f, such that $\lim_{x \to \alpha^+} f(x, \alpha, \beta) = \beta$. (If α or β is $\pm\infty$, it will not be a parameter of f.) Then $f(-, \alpha, \beta)$ is necessarily decreasing on an interval with left endpoint α. If f stops decreasing at some point between α and β, then that point is $A\alpha\beta$-definable. If f does not stop decreasing, then it has a definable infimum (possibly $-\infty$). If that infimum is less than or equal to α, then there must be a fixed point of f that is greater than α and less than β, and that fixed point is $A\alpha\beta$-definable. If the infimum is greater than α, the infimum itself is our desired element. $\square$

Lemma 2.2.8. *Let A be a set, and suppose that, for any $B \supseteq A$, $\mathrm{dcl}(B)$ is dense without endpoints. Then, for any $B \supseteq A$, $\mathrm{Pr}(B) = \mathrm{dcl}(B)$ – in particular, T_A has Skolem functions.*

Proof. It suffices to show that T_A has Skolem functions. Let $\exists x \varphi(x, y)$ be any $L(A)$-formula, x a singleton, such that, for some b a tuple in B, $\mathrm{Pr}(B) \models \exists x \varphi(x, b)$. Then $\varphi(x, b)$ consists of a finite union of intervals and points in $\mathrm{Pr}(B)$. We may definably restrict the domain of y in $\varphi(x, y)$ so that the number of intervals and points is constant, and their relative ordering is always the same. Then, if there are any isolated points in $\varphi(x, b)$, such points are uniformly A-definable from the tuple y, giving a Skolem function. Otherwise, we may take a (uniformly definable) interval satisfying $\varphi(x, b)$, (α, β), with $\alpha, \beta \in \mathrm{dcl}(Ab) \cup \{\pm\infty\}$. Then, by the fact that $\mathrm{dcl}(Ab)$ is dense without endpoints, there must be a point in the interval that is uniformly Ab-definable, which gives us a Skolem function for $\exists x \varphi(x, y)$. $\square$

Lemma 2.2.9. *Let M be a structure, and assume that all elements are non-trivial and that T_M has Skolem functions. Then we may partition I into finitely many sub-intervals (and points) such that, for each subinterval I_i, there is an M-definable binary function with one parameter, $*_x$, such that, for every $x \in I_i$, $*_x$ defines a group chunk on an interval containing x.*

Proof. For each $a \in I$, we know that, in $\bar{M}$, there is a convex group, G, containing a, with the graph of multiplication in G, denoted $*$, given by an $L(M)$-formula, $\varphi(x, y, z, c)$, where $c \in \bar{M}$ is a tuple. We may partition I into finitely many M-definable intervals such that, on each interval, the $L(M)$ formula giving multiplication is the same – if there were infinitely many formulas required, by compactness we could find an element in I for which no formula gave a group chunk. Thus, we may assume that, for all points in I, $\varphi(x, y, z, u)$ gives the graph of multiplication, for some u a tuple in $\bar{M}$.

We may assume that $\varphi(x, y, z, c)$ defines a function on $\bar{M}$, since, for any $b, d \in G$, there must be an isolated point, $g \in G$, such that $\varphi(b, d, g, c)$ holds, so we may assume that it is the only such point, and then that $\varphi(x, y, -, c)$ holds for a single point for any $x, y \in \bar{M}$. Now, note that the properties of a group chunk are first-order, assuming that the function giving the group chunk is definable. Thus, since the properties of a group chunk certainly hold on G, with $*$ the function, the properties of a group chunk must hold on I, for I some $\bar{M}$-definable interval containing a. Then, the formula

$$\psi(a) := \exists w, u_1, u_2(\varphi(x, y, z, w) \text{ defines a group chunk on } (u_1, u_2) \wedge a \in (u_1, u_2))$$

holds for every $a \in I$. Since T_M has Skolem functions, we can find an M-definable function, f, such that $\varphi(x, y, z, f(a))$ defines a group chunk on an interval around a. $\square$

Lemma 2.2.10. *Let c_1, c_2 be noncuts over A, near $\beta_1, \beta_2 \in \mathrm{dcl}(A)$ respectively. If c_1 is not a noncut over c_2A, then there is some A-definable function $f(x)$, such that $\lim_{x \to \beta_1^+} f(x) = \beta_2$ and c_2 lies between $f(c_1)$ and β_2.*

Proof. We assume that c_1, c_2 are above β_1, β_2, respectively – the proof is similar for the other possibilities. Since c_1 is a cut over c_2A, there is some A-definable g such that $\beta_1 < g(c_2) < c_1$. Since $g(c_2)$ cannot be in $\mathrm{dcl}(A)$, we must have $\lim_{x \to \beta_2^+} g(x) = \beta_1$: if not, there is some A-definable interval above β_2 where $\beta_1 < a < g(x)$, for a fixed $a \in \mathrm{dcl}(A)$, which is impossible, since any A-definable interval above β_2 contains c_2, and $\beta_1 < g(c_2) < c_1 < a$ for every $a > \beta_1 \in \mathrm{dcl}(A)$. Thus, $\lim_{x \to \beta_2^+} g(x) = \beta_1$, and as well, g is increasing in a definable neighborhood of β_2 – else we could find an element of $\mathrm{dcl}(A)$ between β_1 and $g(c_2)$. Let $f(x) = g^{-1}(x)$. Then $f(c_1) > c_2$, and moreover $\lim_{x \to \beta_1^+}(f(x)) = \beta_2$. $\square$

Nonuniquely realizable cuts

[Mar86] further categorizes cuts into two kinds.

Definition 2.2.11. Let p be a cut over A. Say p is *uniquely realizable* if, for any (some) $c \models p$, $\mathrm{Pr}(A \cup \{c\})$ has exactly one realization of p. Say p is *nonuniquely realizable* if it is not uniquely realizable.

Example 2.2.12. Let $M = (\mathbb{Q}^{\mathrm{rc}}, +, \cdot, <)$, the ordered field of algebraic real numbers. Then $\mathrm{tp}(\pi/M)$ is a uniquely realizable cut, since $\mathbb{R}$, into which $\mathrm{Pr}(M \cup \{\pi\})$ certainly embeds, has only one realization of the type. On the other hand, let ϵ be an infinitesimal with respect to $\mathbb{Q}$ – in other words, a noncut to the right of 0 over $\mathbb{Q}$, and let $M = (\mathrm{Pr}(\mathbb{Q} \cup \{\epsilon\}), +, <)$, the ordered group generated by $\mathbb{Q} \cup \{\epsilon\}$. Then the type extending the set of formulas $\{x > n\epsilon \mid n \in \mathbb{N}\} \cup \{x < 1/n \mid n \in \mathbb{N}\}$ is a nonuniquely realizable cut, since, if c realizes it, so does $c + \epsilon$, and $c + \epsilon$ must be in the prime model, since the prime model is a group.

Lemma 2.2.13. *Let c realize the type of a uniquely realizable cut over A, and d the type of a nonuniquely realizable cut over A. Then there is no A-definable function, f, such that $f(c) = d$ (and thus no A-definable function such that $f(d) = c$).*

Proof. This is Lemma 3.6 of [Mar86]. □

Lemma 2.2.14. *Let M be a structure, with every element non-trivial, and with T_M having Skolem functions. Let c realize the type of a uniquely realizable cut over M. Suppose there is an M-definable interval, I, around c such that all points in I are non-trivial. Then there is a* $\mathrm{Pr}(M)$*-definable group chunk that contains c.*

Proof. We know that, for every point $b \in I$, there is an Mb-definable group chunk containing b, by Lemma 2.2.9. Let the upper boundary of this group chunk be given by $f(b)$, where f is M-definable, and similarly the lower boundary given by $g(b)$, with g M-definable. Restrict I to a subinterval around c such that both f and g are monotone and continuous. We may assume that, if $x \in (g(y), y)$, then $y \in (x, f(x))$, since we can replace $g(y)$ by $\max(g(y), \inf\{x \mid f(x) > y\})$, with the inf set non-empty, since f would then not be continuous or not be monotone at y. Suppose that, for any $b \in M$ with $b < c$, we have $f(b) < c$. Since $g(c) < c$, and $\mathrm{tp}(g(c)/M) \neq \mathrm{tp}(c/M)$ (since $\mathrm{tp}(c/M)$ is uniquely realizable), we know that there is some $b \in M$ with $b \in (g(c), c)$. But then $f(b) > c$, contradiction. □

Lemma 2.2.15. *Let M be a structure, let c realize the type of a uniquely realizable cut over M, and let f be an M-definable function. Then, for any $a > f(c)$ with $a \in M(c)$, there is $c' \in M$ such that $f(c') \in [f(c), a)$, and similarly if $a < f(c)$.*

Proof. If f is constant in a neighborhood of c, then the lemma is trivial, so assume not. We know that $\mathrm{tp}(f(c)/M)$ is a uniquely realizable cut, since it is interdefinable with c over M. Choose I, a M-definable interval around c such that f is monotonic and continuous on I. Let $a \in M(c)$ with $a > c$. Since $\mathrm{tp}(c/M)$ is uniquely realizable, there is some $a' \in M$ with $c < a' < a$. Then $f^{-1}(a')$ is our desired c'. The case $a < f(c)$ is precisely analogous. □

Lemma 2.2.16. *Let c realize the type of a nonuniquely realizable cut over A, and let d be any element. Then c is a nonuniquely realizable cut over* $\mathrm{Pr}(Ad)$ *iff* $\mathrm{Pr}(Ad)$ *has no realizations of* $\mathrm{tp}(c/A)$.

Proof. First, note that $\mathrm{tp}(c/Ad)$ must be a cut, since otherwise $\mathrm{Pr}(Ad)$ would have to realize $\mathrm{tp}(c/A)$. As well, we know that, for some A-definable function, f, $\mathrm{tp}(f(c)/A) = \mathrm{tp}(c/A)$, with $f(c) \neq c$. Similarly, $\mathrm{tp}(f(c)/Ad)$ must be the same as $\mathrm{tp}(c/A)$, since otherwise, again, $\mathrm{Pr}(Ad)$ would realize $\mathrm{tp}(c/A)$. Thus, f continues to witness that c realizes a nonuniquely realizable cut. □

When is a cut nonuniquely realizable? In order to have multiple realizations in the prime model, there must be a function (definable over the base set), which, when applied to a realization of the cut gives another realization of the cut. In this case, we say that the function *witnesses* the nonuniquely realizableness of the cut. In many cases, we need only consider a restricted set of potential witness functions:

Lemma 2.2.17. *Let T expand the theory of an ordered group, and A be a set. If* $\mathrm{tp}(c/A)$ *is a nonuniquely realizable cut, then, for some A-definable ρ,* $\mathrm{tp}(c/A) = \mathrm{tp}(c+\rho/A)$*. We will use $\rho(c, A)$ to denote such an element. Note that, despite this notation, $\rho(c, A)$ is definable just from A.*

Proof. Since $\mathrm{tp}(c/A)$ is a nonuniquely realizable cut, $\mathrm{dcl}(cA)$ includes another realization of $\mathrm{tp}(c/A)$ besides c. Let $f(c)$ be that realization, where f is A-definable. We may assume that f is monotonic – increasing, otherwise consider f^{-1} – and has no fixed points on an A-definable interval around c. Shrinking the interval further, we can guarantee that there are no fixed points of f in the closure of the interval. Then we can consider the function $f(x) - x$ on the interval. It has a non-zero infimum, which is A-definable, since f is. Call this infimum ρ. Then we can replace the function $f(x)$ by the function $x + \rho$. Since $c < c + \rho \leq f(c)$, we have $\mathrm{tp}(c + \rho/A) = \mathrm{tp}(c/A)$. □

Thus, whenever our theory expands that of an ordered group, we may take the witness function (to a cut being nonuniquely realizable) to be addition by a definable constant.

Lemma 2.2.18. *Let T expand the theory of an ordered group. Let B be a set, $A \subset B$. If* $\mathrm{tp}(c/B)$ *is a nonuniquely realizable cut, but* $\mathrm{tp}(c/A)$ *is not, then some element of* $\mathrm{dcl}(B)$ *is a noncut over A.*

Proof. Note that if $\mathrm{tp}(c/A)$ is a noncut, then we are done, since $\mathrm{dcl}(B)$ must include an element in that type in order to make $\mathrm{tp}(c/B)$ a cut. Thus, we may assume $\mathrm{tp}(c/A)$ is a uniquely realizable cut. Since $\mathrm{tp}(c/B)$ is a nonuniquely realizable cut, we have some B-definable positive ρ such that $\mathrm{tp}(c+\rho/B) = \mathrm{tp}(c/B)$, and, a fortiori, $\mathrm{tp}(c+\rho/A) = \mathrm{tp}(c/A)$. But then ρ is not definable in A, and moreover, no element in $(0, \rho)$ can be definable in A, so ρ is a noncut over A. □

Scales

We can actually further categorize nonuniquely realizable cuts.

Definition 2.2.19. Let $A \subseteq B$ be sets. Let p be a nonuniquely realizable cut over B, with $c \models p$. We say that p is *in scale on A* if, for some A-definable function, $f(x, y)$, with x a tuple and y a singleton, and some tuple $b \in B$, $f(b, \mathrm{dcl}(A))$ is cofinal and coinitial at c in $\mathrm{dcl}(B)$. Say $\mathrm{tp}(c/B)$ is *near scale on A* if there is a function and tuple, as before, such that $f(b, \mathrm{dcl}(A))$ is cofinal (or coinitial) at c in $\mathrm{dcl}(B)$. Say $\mathrm{tp}(c/B)$ is *out of scale on A* otherwise.

Example 2.2.20. Let T be the theory of a real closed field, let $A = \mathbb{Q}^{\mathrm{rc}}$, and let $B = A(\epsilon)$, where ϵ is a noncut above 0 over A. Let $p = \mathrm{tp}(\pi\epsilon/B)$. Then p is in scale on A, since the function $f(\epsilon, y) := y\epsilon$ is cofinal and coinitial at $\pi\epsilon$ in B.

Now, let $A = \mathbb{R}$, with $B = A(\epsilon)$. Let $q(x) \in S_1(B)$ be the complete type saying that $x < a\epsilon$, for $a \in A$, but $x > \epsilon^d$, for $d \in \mathbb{Q}$, $d > 1$. It is not hard to see that q is consistent. (See Example 6.1.5 for the details.) Then q is near scale on A, since the same f is coinitial at any realization of q, but there is no cofinal function. Finally, let $r = \mathrm{tp}(\epsilon^{\sqrt{2}}/B)$ – define this by expanding our language to include exponentiation, taking the prime model of $A\epsilon$ in the new language, yielding the element $\epsilon^{\sqrt{2}}$, then taking the type of this element in the reduct to the original language. Then r is out of scale on A.

Lemma 2.2.21. *Let $p \in S_1(B)$, with $c \models p$, and $A \subseteq B$. If $f(x)$ is a B-definable function such that $f(\mathrm{dcl}(A))$ is cofinal (coinitial) at c in $\mathrm{dcl}(B)$, and $A \subseteq D \subseteq B$, then $f(\mathrm{dcl}(D))$ is cofinal (coinitial) at c in $\mathrm{dcl}(B)$.*

Proof. Trivial. $\square$

Corollary 2.2.22. *Let $p \in S_1(B)$, with $c \models p$, and $A \subseteq D \subseteq B$. If p is in scale on A, it is in scale on D. If p is near scale on A, it is not out of scale on D.*

Lemma 2.2.23. *Let T expand the theory of an ordered group. If p is a nonuniquely realizable cut over B, $c \models p$, and $\mathrm{Pr}(Bc)$ realizes no noncuts over A, then p is in scale on A.*

Proof. Let d be any element of $\mathrm{dcl}(B)$, WLOG greater than c. If $(c, d) \cap \mathrm{dcl}(A) = \emptyset$, then by definition d is a noncut above c over A, and, after subtraction by c, $d - c$ is a noncut above 0. Since $\mathrm{Pr}(Bc)$ realizes no noncuts, $(c, d) \cap \mathrm{dcl}(A)$ is not empty, so $\mathrm{dcl}(A)$ is coinitial at c. Thus, with f the identity, $f(\mathrm{dcl}(A))$ is coinitial at c, and by the symmetric argument, cofinal at c, so p is in scale on A. $\square$

Chapter 3

Bounding Growth Rates

3.1 Previous Work

A good deal of work has been done on various bounds of growth rates of functions definable in o-minimal structures.[1] For instance, [MS98] and [Mil96] give strong bounds on the growth rates of functions definable in o-minimal structures extending groups and fields, respectively. [MS98] shows that, if M is an o-minimal expansion of an ordered group, then either M defines an operation that turns M into a real closed field, or every M-definable function is bounded by an automorphism of M – bounded means that, for sufficiently large values, the automorphism is greater than the function. [Mil96] shows that, if M is an o-minimal expansion of an ordered field, then either every definable function is power-bounded, or the field defines the exponential.

Here, we focus on a result of Miller and van den Dries. In [vdDM96], they show that, given an o-minimal structure expanding a field, the growth of a function definable in any elementary extension of a structure is bounded by a function definable in the original structure. We give a version of their proof:

Proposition 3.1.1. *Let M be an o-minimal structure expanding a real closed field, and let N elementarily extend M, with f an N-definable unary function. Then there exists g, an M-definable unary function, such that, for sufficiently large x, $f(x) \leq g(x)$.*

Proof. Note that, if f is N-definable, we may write it as $f(x, b)$, where b is a finite tuple from N and $f(x, y)$ is M-definable. Thus, we may assume that $N = M(b)$, and thus that N is a finitely generated extension of M. We can then reduce to the case where N is an extension by a single element over M (for the general case, we just apply the result for a single element repeatedly). Thus, we have an M-definable binary function $f(y, x)$, and $a \in N \setminus M$. If $\mathrm{tp}(a/M)$ is a cut, then we can restrict to a cell containing a in its first coordinate and unbounded in its second coordinate on which $f(y, x)$ is monotonic. Then if we choose $d_1 < a < d_2$ with d_1, d_2 in the cell's first coordinate, one of $f(d_1, x)$ and $f(d_2, x)$ must bound $f(a, x)$, since $f(-, x)$ is monotonic.

[1] This topic was originally brought to my attention by [Pet07], in which the growth rates of definable functions are used to define a useful concept called "stationarity," when dealing with groups definable in an o-minimal structure.

Thus, we may assume $\operatorname{tp}(a/M)$ is a noncut. Since T expands the theory of an ordered field, we may assume that a is a noncut near $+\infty$. Let C be the M-definable cell as above, such that $f(y, x)$ is monotonic in each coordinate and $\pi_1(C)$ contains a. Note that C must be increasing in its first coordinate. Let $k(y)$ give the lower boundary of the second coordinate of C in terms of the first. Note that, if $\sup\{f(y, x) \mid k(y) < x\}$ is unbounded for some x, then, since this property is first-order, it is unbounded for some $b \in M$, and thus $f(y, b)$ is our desired function. Otherwise, let $g(x) = \sup\{f(y, x) \mid k(y) < x\}$. Then, for x with $x > k(a)$, $g(x) \geq f(a, x)$. □

3.2 Generalizing

While the use of the field structure above is subtle, it is actually a non-trivial use. The proof works because both arguments of f are coming from the same type – the noncut near ∞. If the two are different noncut types, it is not clear that the same method of bounding the set of which g is the sup will work, and so, without a field structure, the proposition remains to be shown. Fortunately, the purported existence of a fast-growing function actually implies enough structure for our purposes.

Notation 3.2.1. For this theorem, we adopt some terminology to ease exposition. "$P(y)$ for y sufficiently close to b" means that there is an interval with endpoint b such that P holds on the interval, with that interval lying to a consistently-chosen side of b.

Theorem 3.2.2. *If M is an o-minimal structure, $N \succ M$, and $f(a, y)$ is an M-definable function (with a a tuple from N) such that $\lim_{y \to b^-} f(a, y) = c$, for some $b \in M \cup \{\infty\}$, $c \in M \cup \{\pm\infty\}$, then there is an M-definable g such that $\lim_{y \to b^-} g(y) = c$, and for y sufficiently close to b, $g(y) \in [f(y), c)$ (or $(c, f(y)]$). Similarly if f's domain is to the right of b (and $b \in M \cup \{-\infty\}$).*

Proof. Fix N, f, a, b, c satisfying the conditions of the lemma. We assume that $f(a, y)$ approaches c from below, and that the domain of f lies to the left of b. These assumptions do not affect the proof, but allow us to avoid considering all four cases.

First, assume that $f(-, y)$ is constant at a for y sufficiently close to b. Then there is an My-definable interval, I_y, such that $f(-, y)$ is constant on I_y for y sufficiently close to b. But then the value of $f(-, y)$ on I_y is My-definable, say by $g(y)$, so we are done.

Thus, we may assume that $f(-, y)$ is not constant at a. We suppose $h(y) \notin (f(a, y), c)$ for every M-definable h and y sufficiently close to b and prove the proposition, yielding a contradiction. For notation, let $p \in S_1(M)$ be the noncut below c. We can use cell decomposition and assume that f is monotone in x and increasing in y on its two-dimensional domain cell, C, which we can assume is defined by $\{\langle x, y\rangle \mid x \in (d_1, d_2) \wedge b > y > k(x)\}$, for some M-definable function k and $d_1, d_2 \in M \cup \{\pm\infty\}$ (with $d_1 < a < d_2$). We may also assume that $f(C) < c$. We can reduce this proof to the preceding one by proving the following.

Claim 3.2.3. $\operatorname{tp}(a/M)$ *and p are interdefinable.*

Proof. $\operatorname{tp}(a/M)$ is a noncut, by the same argument as for the previous proposition. WLOG, say a is a noncut above. By shrinking C, we may assume that a is a noncut above $c_1 \in$

$M \cup \{-\infty\}$. As the underlying order on M is dense, we know that there is $e \in M$ with $c_1 < a < e < c_2$. It is then clear that, for y sufficiently close to b, $f(-, y)$ is decreasing, else $f(e, y) \in (f(a, y), c)$, since $f(-, y)$ is monotone.

If $k(a) \models p$, then k witnesses the interdefinability of $\mathrm{tp}(a)$ and p. Thus, we can assume that $k(x) < m \in \mathrm{tp}(a/M)$, for some $m \in M$, $m < c$. Then, shrinking c_2 if necessary, we may also assume that $m \geq \sup\{k(x) \mid x \in (c_1, c_2)\}$. Now consider the formula $\varphi(y) := \forall z \exists x \in (c_1, c_2)(f(x, y) \in (z, c) \wedge \langle x, y\rangle \in C)$. Assume that, for y sufficiently close to b, $\varphi(y)$ does not hold. Then, for y sufficiently close to b, the set $\{f(x, y) \mid x \in (c_1, c_2)\}$ has a right endpoint, since it is bounded on the right – note that by our assumption on C, this endpoint is less than c. Let $z(y)$ be this (uniformly My-definable) endpoint. But then $z(y) \in (f(a, y), c)$, contradicting our assumption that no M-definable function is greater than or equal to $f(a, y)$ for y sufficiently close to b. Thus, $\varphi(y)$ does hold for y sufficiently close to b. We can then fix $y_0 \in (m, c)$ in M such that $\varphi(y_0)$ holds, and we have an M-definable map, $f(-, y_0)$. Now we show $f(a, y_0) \models p$. For any $e \in M$, we can find $d \in (c_1, c_2) \cap M$ such that $f(d, y_0) > e$, by φ. Since $d > a$ (else a would not be a noncut near c_1) and $f(-, y_0)$ is decreasing, $f(a, y_0) > f(d, y_0) > e$. Thus, $f(a, y_0) \models p$, and so we have an M-definable map between $\mathrm{tp}(a/M)$ and p. $\square$

Now the proof proceeds as before. We have a function, $f(a, -)$, with $\mathrm{tp}(a/M)$ the noncut below c. If $\sup\{f(y, x) \mid k(y) < x < b\}$ has limit c for some x, then, since this property is first order, it has limit c for some $d \in M$, and thus $f(y, d)$ is our desired function. Otherwise, let $g(x) = \sup\{f(y, x) \mid k(y) < x < b\}$. Then, for x with $x \in (k(a), b)$, $g(x) \in [f(a, x), c)$. $\square$

Chapter 4

Maximal Small Extensions

4.1 Introduction

Marker, in [Mar86], defines:

Definition 4.1.1. Let $M \prec N$. Say N is a *small extension of* M if, for any $a \in N$, finite $A \subset M$, $\mathrm{tp}(a/A)$ is realized in M.

The question is asked, if an o-minimal M does not have unboundedly large small extensions, what is the largest cardinality small extension that M can have?

In [Mar86], it is shown that any such maximal small extension can have cardinality at most $2^{|M|}$. The argument uses the fact that there are at most $2^{|M|}$ types over M. Since there are actually at most $\mathrm{Ded}(|M|)$ types over M, where $\mathrm{Ded}(\alpha) = \sup\{|\bar{Q}| : Q \text{ a linear order}, |Q| \leq \alpha\}$, [Mar86]'s argument shows that a maximal small extension must have cardinality at most $\mathrm{Ded}(|M|)$.

Most examples of o-minimal structures either have no small extensions or unboundedly many – in a pure dense linear order, every extension is small, since any type over finitely many elements is realized – the model is $\aleph_0$-saturated. In the rationals as an ordered group, no extension is small, since every element is $\emptyset$-definable, so any unrealized type was unrealized over $\emptyset$.

In this chapter, we use different notation from the rest of the work. Variables indicate elements of structures, although those elements are often themselves sequences.

4.2 M-Finite Types

[She78] defines the following:

Definition 4.2.1. $p \in S(A)$ is a $\mathbf{F}^s_\lambda$-type if $|A| < \lambda$. Say $p \in S(A)$ is a $\mathbf{F}^s_\lambda$-type if, for some $B \subseteq A$, $|B| < \lambda$, there is $q \in S(B)$ such that $q \vdash p$.

If $p \in S(M)$, $A \subseteq M$, p is a $\mathbf{F}^s_{\aleph_0}$-type witnessed by some finite subset of A, we will call p *A-finite*. In other words,

Definition 4.2.2. Given a model, M, and a set, $A \subseteq M$, let a type $p \in S_1(M)$ be *A-finite* iff for some finite $\bar{b} \in A$, $p \restriction \bar{b}$ generates p. Say p is *almost A-finite* iff for some type, q, definable from p, q is A-finite.

Example 4.2.3. If $M = (\mathbb{Q}, +, <, 0)$, $N = M(\pi)$, and $p \in S_1(N)$ is a noncut above π, then p itself is not M-finite, but, if $c \models p$, $c - \pi$ is M-finite.

Shelah's interest in **F**-types was in constructing prime models, so in realizing only **F**-types. Here, the opposite is true: if an extension is small, then it realizes no M-finite types.

Since order-type implies type in o-minimal theories, A-finiteness has an interpretation in the order – $\mathrm{dcl}(\bar{b})$ is dense around p, for $\bar{b} \subseteq A$ the witness to A-finiteness. Considering this interpretation, we see that if M is o-minimal, realizing no M-finite types implies that an extension is small.

4.3 Existence of Maximal Small Extensions

Theorem 4.3.1. *For every α, there is an o-minimal structure, M, $|M| = \alpha$, with small extensions but not unboundedly large small extensions. Moreover, if α is of the form $\beta^{<\lambda}$, for some λ, a small extension can be found of cardinality β^{λ}.*

Proof. We give a construction of models M and N, with $M \prec N$, and N a maximal small extension of M. We then verify the sizes of M and N.

Let G be a divisible ordered abelian group, λ an ordinal, Q a dense divisible proper subgroup of G. Let $Q' = G \setminus Q$.

Let $M = G^{<\lambda}$, ordered lexicographically and equipped with group structure component-wise. Let our language be that of an ordered group, extended by constants for every element of $Q^{<\lambda}$. We will build N in stages.

- Let $M_0 = M$.
- Given M_i, choose $a \in G^{\lambda}$ such that any $b \in \mathrm{dcl}(aM_i) \setminus M_i$ has cofinally many components in Q'. Let $M_{i+1} = M_i(a)$. Take unions at limits.

This construction must halt at some point, since there are $\leq |G|^{\lambda}$ elements to add. Let the union of the M_is be N.

The original M is o-minimal, since it is a divisible ordered abelian group, and each M_i and N is an elementary extension, since it is also a divisible ordered abelian group, and this theory has quantifier elimination.

It remains to be shown that N is a small extension of M, and that there is no larger small extension of M. In fact, we show that every small extension of M comes from this type of construction.

Notation: we use M' to denote an arbitrary M_i or N. As well, for $\alpha < \lambda$, $a[\alpha]$ is the αth component of a, and $a \restriction \alpha = \langle a[i] \rangle_{i<\alpha}$.

Lemma 4.3.2. *Every noncut over M' is almost M-finite.*

Proof. Let p be a noncut over M'. Assume p is of the form $\{a < x\} \cup \{x < e \mid e \in M', e > a\}$. (The case $x < a$ is precisely symmetric. If p is near $\pm\infty$, then p is M-finite, since $Q^{<\lambda}$ is cofinal in G^λ.) Let d be any realization of p. The type of $d - a$ over M' is generated by $\{0 < x\} \cup \{x < e \mid e > 0\}$ – the noncut near 0. Given any $e > 0 \in M'$, let α be the first index at which $e[\alpha] \neq 0$. Let $c \in Q^{\alpha+1}$ be such that $c[i] = 0$ for $i < \alpha$, and $0 < c[\alpha] < e[\alpha]$. We know $0 < c < e$. Thus, $x < c$ implies $x < e$, and $d - a < c$, so $\mathrm{tp}(d - a/M')$ is generated by $\mathrm{tp}(d - a)$.

□

Definition 4.3.3. If $p \in S_1(M')$ is a cut, and there is some $\alpha < \lambda$ such that, for $a, b \in M'$, $a \restriction \alpha = b \restriction \alpha$ implies $x < a \in p \iff x < b \in p$, then p is *reducible.*

Note that if p is reducible, then it is not uniquely realizable.

Lemma 4.3.4. *If p is reducible, then p is M-finite.*

Proof. Let α be the least such in the definition of reducible. For each $\beta < \alpha$, we can find $a_\beta \in M$, a_β^+, a_β^- extending a_β such that $\mathrm{lh}(a_\beta) = \beta$, $x < a_\beta^+ \in p$, and $x > a_\beta^- \in p$. It is easy to see that $\beta < \beta'$ implies a_β is an initial segment of $a_{\beta'}$. Let $a = \bigcup_{\beta<\alpha} a_\beta$, so $a \in M$.

Let d realize p, let e be any element of M'. WLOG, assume $e > d$. We show $e > d$ is implied by $\mathrm{tp}(d/a)$.

Case 1: $e \restriction \alpha \neq a \restriction \alpha$. Then e and a differ at some coordinate $\beta < \alpha$, so $e[\beta] > a[\beta]$.

If $a > d$, we are done. Otherwise, by density of Q, we can find $c \in Q^{\beta+1}$ with $c[i] = 0$ for $i < \beta$, and $a[\beta] < a[\beta] + c[\beta] < e[\beta]$. Again, it is clear that $a + c > d$, so we are done for this case.

Case 2: $e \restriction \alpha = a \restriction \alpha$. Since we assume $e > d$, we also have $a > d$, by definition of p. Let $\beta \geq \alpha$ be the first coordinate at or past α at which e is not 0 (otherwise $e = a$). If $e[\beta] > 0$, we are done, so let $e[\beta] < 0$. Choose $c \in Q^{\beta+1}$ such that $c[i] = 0$ for $i < \beta$, and $c[\beta] < e[\beta] < 0$. Then $c + a < e$, but since $(c + a) \restriction \alpha = e \restriction \alpha$, $c + a > d$, which implies $e > d$. □

Lemma 4.3.5. *If $p \in S_1(M')$ is a non-reducible cut, then for some $a \in G^\lambda$,* $\mathrm{tp}(a/M') = p$.

Proof. For each $\alpha < \lambda$, by non-reducibility, there are $a_\alpha^-, a_\alpha^+ \in M'$ such that $a_\alpha^- \restriction \alpha = a_\alpha^+ \restriction \alpha$, but $a_\alpha^- < x < a_\alpha^+ \in p$.

Let $a_\alpha = a_\alpha^- \restriction \alpha$. It is easy to check that $\alpha < \alpha'$ implies $a_\alpha \subset a_{\alpha'}$.

Let $a = \bigcup_{\alpha<\lambda} a_\alpha$. If $a < e$, then at some component, say α, $a[\alpha] < e[\alpha]$. But $a \restriction \alpha + 1 = a_{\alpha+1}^+$, so $a_{\alpha+1}^+ < e$, so $x < e \in p$.

The case $e < a$ is symmetric. Thus, $\mathrm{tp}(a/M') = p$. □

Lemma 4.3.6. *Let $d \in G^\lambda$ realize a non-reducible cut over M' without cofinally many components in Q'. Then* $\mathrm{tp}(d/M')$ *is M-finite.*

Proof. For some $m < \lambda$, $b = d \restriction m$ has all the components of d in Q'. Note that $b \in M$. Given any $e \in M'$, if $x < e$ is in $\mathrm{tp}(d/M')$, then let n be the first index at which d and e differ.

If $n < m$, let $c \in Q^{n+1}$ be such that $c[i] = 0$ for $i < n$, and $0 < c[n] < e[n] - b[n]$. Then $x < b + c$ is in $\mathrm{tp}(d/b)$, and $b + c < e$.

If $n \geq m$, then choose $c \in Q^{n+1}$ such that $c[i] = 0$ for $i < m$, $c[i] = d[i]$ for $m \leq i < n$, and $d[n] < c[n] < e[n]$. Then $x < b + c$ is in $\mathrm{tp}(d/b)$ and $b + c < e$.

The $e < x$ case is symmetric. $\square$

Lemma 4.3.7. *If $d \in G^\lambda \setminus M'$ has cofinally many Q' components, then $\mathrm{tp}(d/M')$ is not M-finite. Thus, if every $b \in \mathrm{dcl}(dM') \setminus M'$ has cofinally many components in Q', then d is not almost M-finite.*

Proof. Assume for a contradiction that $\mathrm{tp}(d/M')$ is M-finite. Let $\bar{b} = (b_1, \ldots, b_m)$ witness this, of minimal length (as a tuple).

For any $a \in M'$, we can find $f(\bar{b})$, with f $\emptyset$-definable, such that $f(\bar{b})$ lies between d and a. Considering $d \restriction i$, for $i < \lambda$, we can find $\{f_i(\bar{b})\}_{i<\lambda}$ with $f_i(\bar{b}) \restriction i = d \restriction i$.

By q.e. for divisible ordered abelian groups, we know that each $f_i(\bar{b})$ is an affine linear combination (with rational coefficients) of the b_j's, with the affine part given by $c \in Q^{<\lambda}$. If we take $\alpha = \max(\mathrm{lh}(b_j) \mid j \leq m)$, then for any β, $f_\beta(\bar{b})$ can have no Q' components past the αth one. But this is clearly impossible. $\square$

This completes our proof that N is a maximal small extension of M. We know that N is a proper extension of M, since any element with cofinal Q' components can be adjoined to form M_1. It remains to determine its size. We lose nothing by restricting to the case where λ is an infinite cardinal.

Any element of $\mathrm{dcl}(M_i a) \setminus M_i$ can be written as $q(a+b)$, where $q \in \mathbb{Q}$, and $b \in M_i$. Since $a + b$ has cofinal Q' components iff $q(a + b)$ does, we need only consider $a + b$, for $b \in M_i$.

We can then rephrase in the terminology of vector spaces: M is a subspace of G^λ as a $\mathbb{Q}$-vector space. We wish to find linearly independent $\{a_i\}_{i<\beta} \in G^\lambda$ such that $(M + Q^\lambda) \oplus \mathrm{span}(\{a_i\}_{i<\beta}) = G^\lambda$.

Let W be a subgroup of G such that $G = Q \oplus W$. Let $\gamma = \dim W$. Then $G^\lambda = Q^\lambda \oplus W^\lambda$, and $\dim W^\lambda = (\gamma^+)^\lambda$ FIXME $(\gamma+1?)$. Moreover, we can write $W^\lambda = W^{<\lambda} \oplus X$, for some X, and $M + Q^\lambda = W^{<\lambda} + Q^\lambda$. Thus, $\beta = \dim X$.

Claim 4.3.8. $\dim X = \gamma^\lambda$.

Proof. We construct a set of independent (even including $W^{<\lambda}$) elements of W^λ, with size γ^λ, each of length λ, showing that $\dim X \geq \gamma^\lambda$, which is enough.

Since $\lambda \times \lambda = \lambda$, we can find λ disjoint subsets of λ of length λ (necessarily cofinal). Let $\{X_i \mid i < \lambda\}$ be the characteristic functions of these subsets – each X_i is a binary sequence of length λ.

Since $\dim W = \gamma$, it has a basis of size γ, $\{b_i\}_{i<\gamma}$. For $b \in W$, let bX_i denote the element of W^λ obtained by replacing each 1 in the sequence X_i by b.

For $f \in W^\lambda$, let $A_f = \sum_{i<\lambda} f(i)X_i$. This sum is well-defined, because no two X_is are non-zero on the same component. We know that there is a basis of W^λ of size γ^λ, say $\{f_i\}_{i<\gamma^\lambda}$. Denote A_{f_i} by A_i. We show that $\{A_i \mid i < \gamma^\lambda\}$ is linearly independent and its span is disjoint from $W^{<\lambda} \setminus \{0\}$. WLOG, it is enough to show that no non-zero linear combination of $A_1, \ldots, A_n$ is in $W^{<\lambda}$.

Suppose that $q_1 A_1 + \ldots + q_n A_n = c$, where $q_i \in \mathbb{Q}$, $c \in W^{<\lambda}$. This then implies that $\sum_{i<\lambda}(\sum_{j\leq n} q_j f_j(i))X_i = c$. If $k, l \in X_i$, then it is clear that the left-hand side has the same value at its k and l component, so in fact $c = 0$ (choose $k < \mathrm{lh}(c) < l$). But this means precisely $\sum_{j\leq n} q_j f_j = 0$, and so $q_j = 0$, $j \leq n$, and hence the A_is are linearly independent.

□

Now we have that, for W such that $G = Q \oplus W$ with $\dim W = \gamma$, $|N| = |M| + \gamma^\lambda$.

For any α, an elementary compactness argument shows there exist G and Q such that $|G| = |W| = \alpha$, so we can take $\gamma = |G|$, and so $|N| = \gamma^\lambda$.

□

When $\lambda = \omega$, $G = \mathbb{Q}^{\mathrm{rc}}$, and $Q = \mathbb{Q}$, then $|M| = \aleph_0$ and $|N| = 2^{\aleph_0}$: the bound is as sharp as possible. In general, we can take G to have cardinality α, and λ to be ω. Then $|M| = \alpha$. However, while N exists, it is possible that $|N| = |M|$.

Chapter 5

Decreasing Types

5.1 Definition and Basic Properties

Given an n-type, the ordering of the variables can affect the type of each variable over the preceding one. For instance, consider the type of (π, ϵ) over $M = (\mathbb{Q}^{\mathrm{rc}}, +, \cdot, <)$, where ϵ is an infinitesimal. We have $\mathrm{tp}(\pi/M)$ is a uniquely realizable cut, while $\mathrm{tp}(\epsilon/\pi M)$ is a noncut. However, if we consider the elements in reverse order, $\mathrm{tp}(\epsilon/M)$ is still a noncut, but now $\mathrm{tp}(\pi/M\epsilon)$ is a nonuniquely realizable cut. We wish to fix a class of orderings of p's coordinates that will provide some predictability in the cuts and noncuts.

Convention 5.1.1. In this chapter, we will assume that T is such that all noncuts are interdefinable over the empty set, except where otherwise noted. By Lemma 2.2.7, this implies that there is at least one $\emptyset$-definable element. As well, by Lemma 2.2.5, around this $\emptyset$-definable element is an $\emptyset$-definable group chunk, with an $\emptyset$-definable identity element. Let "0" denote some such $\emptyset$-definable element such that there exists an $\emptyset$-definable group chunk containing it, and in which it is the identity element.

We begin by defining a partial ordering that we will use henceforth.

Definition 5.1.2. Let A be a set. Define $a \prec_A b$ iff there exists $a' \in \mathrm{dcl}(aA)$ such that $a' > 0$, and $(0, a') \cap \mathrm{dcl}(bA) = \emptyset$. Define $a \sim_A b$ if $a \not\prec_A b$ and $b \not\prec_A a$. Finally, let $a \precsim_A b$ if $a \sim_A b$ or $a \prec_A b$.

Lemma 5.1.3. *$\sim_A$ is an equivalence relation, and $\prec_A$ totally orders the $\sim_A$-classes.*

Proof. It is trivial to see that $\sim_A$ is an equivalence relation – transitivity is true because coinitiality (near 0) is transitive. Similarly, $\prec_A$ totally orders the $\sim_A$-classes because "coinitiality" totally orders sets, up to coinitiality equivalence. ☐

Lemma 5.1.4. *Let A be a set, and suppose that a, b are noncuts near $\alpha \in \mathrm{dcl}(A) \cup \{\pm\infty\}$. Then, if $a \in (\alpha, b)$ (or $a \in (b, \alpha)$, if $b > \alpha$), $a \precsim_A b$.*

Proof. We assume $b > \alpha$ for simplicity. Let $c \in \mathrm{dcl}(bA)$. If c is not a noncut above 0 over A, then it does not pose a problem, so we may assume that c is a noncut above 0 over A. We may write $c = f(b)$, where f is A-definable, and f is necessarily non-constant in a neighborhood above α. By Lemma 2.2.2, f is increasing, so $f(a) < f(b) = c$. Thus, no element of $\mathrm{dcl}(bA)$ is a noncut near 0 over $\mathrm{dcl}(aA)$, showing that $a \precsim_A b$. ☐

Lemma 5.1.5. *Let $A \subset B$ be sets, and let c, d be noncuts above 0 over B, and let $c \sim_B d$. Then $c \sim_A d$.*

Proof. Suppose not. WLOG, assume $c < d$. We know that both c and d are noncuts above 0 over A. Then, by Lemma 5.1.4, $c \precsim_A d$. If $c \prec_A d$, then for some A-definable function, f, $f(c)$ is a noncut above 0 over Ad. Considering $f(\mathrm{dcl}(Ad))$, we see that c is a noncut above 0 over Ad. Since c is not a noncut above 0 over Bd, $\mathrm{Pr}(Bd)$ must realize a noncut above 0 over Ad. Let $g(d)$ be such a noncut, with g a B-definable function. But we know, by Theorem 3.2.2, that there is an A-definable function, h, such that, for sufficiently small x, $0 < h(x) < g(x)$. Since d is a noncut near 0 over both B and A, d is certainly sufficiently small, so $h(d) < g(d)$. But then $h(d) < c$, contradiction. $\square$

Definition 5.1.6. If $A \subset B$, and b is an element, we say that b is *$\prec_A$-maximal over B* if $b \succsim_A c$ for every $c \in B \setminus A$.. Similarly for strictly $\prec_A$-maximal, and for -minimal. Given a sequence, $c = \langle c_i \rangle_{i \in I}$, with I an ordered set, and $J \sqsubseteq I$, we say that b is *J-maximal* if b is $\prec_{c_{\leq J}}$-maximal over c. Similarly for strictly J-maximal, and for -minimal.

Lemma 5.1.7. *Let A be a set, and let b realize a cut over A. Then b is $\prec_A$-maximal over B, for any set B.*

Proof. Note that $\mathrm{dcl}(Ab)$ does not realize a noncut above 0 over A, since b is a cut. Thus, $\mathrm{dcl}(A)$ is coinitial at 0 in $\mathrm{dcl}(Ab)$. Therefore, since, for any $c \in B$, $\mathrm{dcl}(Ac) \supseteq \mathrm{dcl}(A)$, we can never have $(0, a) \cap \mathrm{dcl}(Ac) = \emptyset$ for any $a \in \mathrm{dcl}(Ab)$. $\square$

Lemma 5.1.8. *Let $A \subseteq B$ be sets, and suppose that $\mathrm{tp}(b/B)$ is a noncut near some element of $\mathrm{dcl}(A) \cup \{\pm\infty\}$. Then b is strictly $\prec_A$-minimal over B.*

Proof. By assumption of interdefinability of noncuts over $\emptyset$, we know that there is an α-definable function sending b to a noncut above 0 over B, which suffices. $\square$

Lemma 5.1.9. *Let $A \subset B$ be sets, and let b be a strictly $\prec_A$-maximal element over B. Then b is not a noncut near any $a \in \mathrm{dcl}(B) \setminus \mathrm{dcl}(A)$ over Aa.*

Proof. Let $N = \mathrm{dcl}(B)$, $M = \mathrm{dcl}(A)$. Suppose not, and let $a \in N \setminus M$ be an element near which b is a noncut over Ma. We show that $a \sim_M b$, contradicting b's strict maximality. If b is a cut over M, then a is also a cut over M, contradicting strict maximality, so we may assume that b is a noncut over M. Let $f(b)$ be a noncut above 0 over M, with f an M-definable function non-constant in a neighborhood of b. If some element of $M(a)$ lies between $f(a)$ and $f(b)$, f^{-1} of that element would lie between a and b, so $f(b)$ is still a noncut near $f(a)$ over Ma, so we may replace b by $f(b)$ and a by $f(a)$, and assume that a is a noncut above 0 over M and b is a noncut above a over Ma (and hence a noncut above 0 over M). We know that $(0, a)$ must have an element of $\mathrm{dcl}(Ma)$ in it, say $f(a)$, where f is M-definable. The function f must be increasing in a neighborhood of 0, with $f(x) < x$, else a could not be a noncut near 0. Thus, $f(b) \in (0, b)$. If $f(b) > a$, then $f^{-1}(a) < b$. But then b cannot be a noncut near a over Ma, since $f^{-1}(a) \in M(a)$. Thus, $f(b) < a$. Thus, by Lemma 5.1.4, $b \precsim_M a$, contradicting strict maximality of b. $\square$

Lemma 5.1.10. *Let $A \subseteq B$ be sets, and let b be strictly $\prec_A$-maximal over B. Then b is not a noncut over C, for any $A \subseteq C \subseteq B$.*

Proof. By Lemma 5.1.9, we know that, for any such C, b is not a noncut near an element of $\mathrm{dcl}(C) \setminus \mathrm{dcl}(A)$, so b would have to be a noncut near an element of $\mathrm{dcl}(A)$. But then b would not be strictly $\prec_A$-maximal over C, by Lemma 5.1.8. $\square$

Lemma 5.1.11. *Let $A \subset B$, assume $\mathrm{dcl}(B)$ realizes no cuts over A, and let c be an element, with $\mathrm{tp}(c/B)$ near scale on A. Then $\mathrm{dcl}(Bc)$ contains an element that is strictly maximal in the $\prec_A$-ordering over $\mathrm{dcl}(B) \setminus \mathrm{dcl}(A)$.*

Proof. Since $\mathrm{tp}(c/B)$ is near scale on A, there is some B-definable function, f, such that $f(\mathrm{dcl}(A))$ is (WLOG) cofinal at c in $\mathrm{dcl}(B)$. If f has a constant value below c, then c must be a noncut over B near this constant value, contradicting the fact that it is a nonuniquely realizable cut. Thus, f is not constant, and so must have image including c. Consider $f^{-1}(c)$. If $\mathrm{tp}(f^{-1}(c)/A)$ is a cut, then we may take b an element of B such that $b > c$ and f^{-1} is continuous and monotonic on an interval containing (c, b). But then, since $\mathrm{tp}(f^{-1}(c)/A) = \mathrm{tp}(f^{-1}(b)/A)$, we have that $f^{-1}(b)$ is a cut over A, but this contradicts our assumption that $\mathrm{dcl}(B)$ realizes no cuts over A. Thus, $f^{-1}(c)$ is a noncut over A. We may assume that it is a noncut above 0. I claim that $c' = f^{-1}(c)$ is strictly maximal in the $\prec_A$-ordering over $\mathrm{dcl}(B) \setminus \mathrm{dcl}(A)$. Suppose not, so let $b \in \mathrm{dcl}(B) \setminus \mathrm{dcl}(A)$ be such that $b \succsim_A c'$. Since $\mathrm{dcl}(B)$ realizes no cuts over A, b is a noncut over A. Since $b \succsim_A c'$, there is some A-definable function, g, such that $g(b)$ is a noncut above 0 over A, but $g(b) > c' > 0$. Replace b by $g(b)$. But then $f(g(b))$ contradicts that $f(A)$ is cofinal at c in $\mathrm{dcl}(B)$. $\square$

Lemma 5.1.12. *Let A be a set and let c be a strictly decreasing sequence over A. Then each c_i is algebraically independent from $Ac_{\neq i}$.*

Proof. We can show this by proving that c_n is independent from $Ac_{<n}$, going by induction. We know that $\mathrm{dcl}(Ac_n)$ includes some point, e, such that $(0, e) \cap \mathrm{dcl}(Ac_{<n}) = \emptyset$. By denseness of definable closures, this means that we have some $f(e) \in (0, e)$, with f an A-definable function, so $f(e) \notin \mathrm{dcl}(Ac_{<n})$, but $f(e) \in \mathrm{dcl}(Ac_n)$, so $c_n \notin \mathrm{dcl}(Ac_{<n})$. $\square$

Lemma 5.1.13. *Let A be a set, and let c be a sequence of length n strictly ordered by $\prec_A$. Then $\mathrm{dcl}(Ac_i)$ is coinitial at 0 in $\mathrm{dcl}(Ac_{\leq i})$, for each $i \leq n$.*

Proof. Suppose not. Let i be minimal witnessing failure. Then $\mathrm{dcl}(Ac_{i-1})$ is coinitial at 0 in $\mathrm{dcl}(Ac_{<i})$, but $\mathrm{dcl}(Ac_i)$ is not coinitial at 0 in $\mathrm{dcl}(Ac_{\leq i})$. Note that $i > 1$. Since $c_i \prec_A c_{i-1}$, we know that c_i is a noncut near $\alpha \in \mathrm{dcl}(A)$ over A. Since $\mathrm{dcl}(Ac_i)$ is not coinitial, we know that there is some $f(c_{\leq i})$, f an A-definable function, such that $f(c_{\leq i})$ is a noncut above 0 over Ac_i.

Claim 5.1.14. *c_i is not a noncut near α over $Ac_{<i}$.*

Proof. $f(c_{<i}, -)$ is an $Ac_{<i}$-definable function. We can find an A-definable g, such that, for x sufficiently close to α, $0 < g(x) \leq f(c_{<i}, x)$. If c_i were a noncut near α over $Ac_{<i}$, then c_i would be "sufficiently close" to α, so it is not. $\square$

Thus, there is some A-definable function, g, such that $g(c_{<i})$ lies between α and c_i. By minimality of i, we know then that there is an A-definable g' such that $g'(c_{i-1})$ lies between α and c_i. But then, by Lemma 5.1.4, $g'(c_{i-1}) \precsim_A c_i$, which implies $c_{i-1} \precsim_A c_i$, which contradicts the strict ordering of c. $\square$

Lemma 5.1.15. *Let A be a set, and let $c_1, \ldots, c_k$ be elements with $c_i \prec_A c_j$ for $i > j$. Then $c_i \prec_j c_j$ for $i > j$.*

Proof. Fix $i > j$. Let $e \in \operatorname{dcl}(Ac_i)$ be such that $(0, e) \cap \operatorname{dcl}(Ac_j) = \emptyset$. By Lemma 5.1.13, $\operatorname{dcl}(Ac_j)$ is coinitial at 0 in $\operatorname{dcl}(Ac_{\leq j})$. Thus, $(0, e) \cap \operatorname{dcl}(Ac_{\leq j}) = \emptyset$, so $c_i \prec_j c_j$. $\square$

Corollary 5.1.16. *Let A be a set, and let $c = \langle c_1, \ldots, c_k \rangle$ with $c_i \prec_A c_j$ for $i > j$. Then for any A-definable non-constant function, f, $f(c) \sim_A c_i$, for some $1 \leq i \leq k$.*

Proof. Let $f(x_1, \ldots, x_k)$ be any A-definable function. We may assume that f is non-constant on x_k in a neighborhood of $c_{\leq k}$, otherwise we may shorten c and take f as a function in $k-1$ variables.

We first show that $f(c) \succsim_A c_k$. Suppose not. Then $f(c) \prec_A c_k$, and thus $\langle c, f(c) \rangle$ satisfies the conditions of Lemma 5.1.15, and so the conclusion holds, in particular that $f(c) \prec_k c_k$. But this is impossible, because $f(c) \in \operatorname{dcl}(Ac_{\leq k})$.

Now, suppose $f(c) \not\sim_A c_i$, for any $1 \leq i \leq k$. Suppose that $f(c)$ comes before c_j, for some $1 \leq j \leq k$, in the $\prec_A$ order. Then if we consider the tuple $\langle c_1, \ldots, c_{i-1}, f(c), c_i, \ldots, c_{k-1} \rangle$, it satisfies the conditions of the corollary, and so, by what we have just proved, any definable function of this tuple, $g(c_{<i}, f(c), c_i, \ldots, c_{k-1})$ is at least as big as c_{k-1} in the $\prec_A$ ordering. But, by exchange, we can take g so that $g(c_{<i}, f(c), c_i, \ldots, c_{k-1}) = c_k$, contradiction. $\square$

Lemma 5.1.17. *Let A be a set, and let c be a sequence strictly ordered by $\prec_A$. Then each element of c is independent of the others over A.*

Proof. Apply Lemma 5.1.15 and then Lemma 5.1.12. $\square$

Lemma 5.1.18. *Let A be a set, and let c be a tuple, with $\operatorname{lh}(c) = n$. Then $\operatorname{dcl}(Ac) \setminus \operatorname{dcl}(A)$ contains a maximal and a minimal element with respect to the $\prec_A$-ordering. In fact, any chain strictly ordered by $\prec_A$ in $\operatorname{dcl}(Ac)$ has at most n elements.*

Proof. Since the elements of the chain are algebraically independent, and they are all in $\operatorname{dcl}(Ac)$, which is generated by at most n elements over A, there can be at most n elements in the chain by Lemma 1.3.9. $\square$

Definition 5.1.19. Given a base set, A, and a tuple, $c = \langle c_1, \ldots, c_n \rangle$, $c_j \prec_i c_k$, for $i \leq j, k$, iff $c_j \prec_{Ac_{<i}} c_k$. Finally, given a type, $p(x_1, \ldots, x_n)$, let $x_j \prec_i x_k$ iff, for some realization c of p, $c_j \prec_i c_k$.

Note that, in the last part, if some realization of p has this property, then so does every realization of p. In this chapter, we will only consider the case of types in finitely many variables. While the definitions, and most results, are specializations of the general case (in Section 5.3), their exposition is easier in the finite case. Moreover, in Chapter 6 we will only need the finite case.

Lemma 5.1.20. *Let p be an n-type over a set A. Then there exists a re-ordering of the variables of p such that, in the new ordering, $x_j \succsim_i x_k$, for $1 \leq i \leq j < k$.*

Proof. We re-order p in stages. At stage i, having determined $x_{<i}$, there is at least one maximal element in the order $\prec_i$ among the remaining x_j. Set any such maximal element to be x_i. $\square$

Definition 5.1.21. If the variables of p satisfy the conclusion of Lemma 5.1.20, we say that p is *decreasing.* For i an index in the variables of p, let $N(i)$ denote the greatest index at most i such that $\text{tp}(c_{N(i)}/c_{<N(i)}A)$ is a noncut, and 0 if such index does not exist.

Lemma 5.1.22. *Let p be a decreasing type over a set A, let $c \models p$, and let i be an index such that $\text{tp}(c_i/Ac_{<i})$ is a noncut. Then for $k \geq i$, $\text{tp}(c_k/c_{<i}A)$ is a noncut.*

Proof. Note that, since $c_i \succsim c_k$ (by definition of "decreasing"), we know that $\text{dcl}(c_kAc_{<i})$ is coinitial above 0 in $\text{dcl}(Ac_{\leq i}$. Since c_i is a noncut over $Ac_{<i}$, there is some $d \in \text{dcl}(Ac_{\leq i})$, a noncut above 0 over Ac_i. By coiniality, there is some $d' \in \text{dcl}(c_kAc_{<i})$, with $0 < d' < d$, but then d' witnesses that c_k is a noncut over $Ac_{<i}$. $\square$

Lemma 5.1.23. *Let p be a decreasing type over a set A, let $c \models p$, and let $k = N(i) < i$. Then $\text{tp}(c_i/Ac_{\leq k})$ is a cut. Moreover, $\text{Pr}(Ac_{\leq i})$ realizes only cuts over $Ac_{\leq k}$.*

Proof. We show that $N_j = \text{Pr}(c_{\leq j}A)$ realizes only cuts over $M = \text{Pr}(Ac_{\leq k})$, for all $k < j \leq i$, proving the lemma. The case $k+1$ is from the definition of $N(i)$. We know that $\text{tp}(c_j/Ac_{<j}) = \text{tp}(c_j/N_{j-1})$ is a cut, and that N_{j-1} realizes no noncuts. Thus, N_j realizes only cuts over M, by Lemma 2.2.3. $\square$

Lemma 5.1.24. *If c is a decreasing sequence over Ad, and d is maximal in the $\prec_A$-ordering over c, then $\langle d, c\rangle$ is a decreasing sequence over A.*

Proof. Let $c' = \langle d, c\rangle$. By assumption, $c'_1 \succsim_A c'_i$, for all $i > 1$. For $j > 1$, $c'_j \succsim_j c'_i$, for $i > j$, iff $c_{j+1} \succsim_{Adc_{<j+1}} c_{i+1}$, but that is given, since c is a decreasing sequence over Ad. $\square$

Lemma 5.1.25. *Let a be a tuple, with $\text{lh}(a) = n$, and M a structure. Then there is a tuple, $a' \in \text{dcl}(Ma)$, with $\text{dcl}(Ma) = \text{dcl}(Ma')$, such that a' is decreasing, and there is $1 \leq i_1 \leq n$ such that $\text{tp}(a_j/M)$ is a cut iff $j < i_1$, and for $j \geq i_1$, $\text{tp}(a_j/Ma_{<j})$ is not in scale or near scale on M. Moreover, if a'_j is a noncut over $Ma_{<k}$ for any $k \leq j$, then it is a noncut above 0.*

Proof. Go by induction on n. We may reorder a so that it is decreasing and assume by induction that $a_i \notin \text{dcl}(Ma_{<i})$, for $i \leq n$. Let i be an index such that, for some $c \in \text{dcl}(Ma_{\leq i}) \setminus M$, c is a maximal element of the $\prec_M$-ordering over $\text{dcl}(Ma) \setminus M$ – such an element exists by Lemma 5.1.18. By induction, we can find a'' interdefinable with this tuple such that a'' satisfies the conclusions of the lemma over Mc, say with i'_1. We may also assume that, if $\text{tp}(a_i/Ma_{<j})$ for some $j \leq i$ is a noncut, then it is a noncut above 0.

Then $a' = \langle c, a''\rangle$ is decreasing over M by Lemma 5.1.24. Since a'_1 was maximal in the $\prec_M$-ordering, if $\text{tp}(a'_j/M)$ is a cut over M for $j \leq n$, then $\text{tp}(a'_1/M)$ must also be a cut over M by Lemma 5.1.7, and thus if $\text{tp}(a'_j/Ma'_1)$ is a cut, then $\text{tp}(a'_j/M)$ is a cut. It is not possible for $\mathrm{tp}(a'_j/M)$ to be a cut, but $\mathrm{tp}(a'_j/Ma'_1)$ to be a noncut, since if $\mathrm{tp}(a'_j/Ma'_1)$ is a noncut, it is a noncut near 0, and hence a noncut over M. Thus, if $i'_1 > 1$, then we take $i_1 = i'_1 + 1$. If $i'_1 = 1$, then if $\mathrm{tp}(a'_1/M)$ is a cut, take $i_1 = 2$, and otherwise $i_1 = 1$. We now know that, if $j < i_1$, $\mathrm{tp}(a'_j/M)$ is a cut, and if $j \geq i_1$, $\mathrm{tp}(a'_j/M)$ is not a cut. We also know that $\mathrm{tp}(a'_j/Ma'_{<j})$ is not in scale on M, for $j \geq i_1$, since $\mathrm{tp}(a'_j/Ma'_{<j})$ is not in scale on $M(a'_1)$, which is a weaker condition.

We know that $\mathrm{tp}(a'_j/Ma'_{<j})$ is not near scale on Ma'_1. If $\mathrm{tp}(a'_1/M)$ is a cut, then that implies that $\mathrm{tp}(a'_j/Ma'_{<j})$ is not near scale on M, since M is cofinal in $M(a'_1)$. If $\mathrm{tp}(a'_1/M)$ is a noncut, then $\mathrm{tp}(a'_j/Ma'_{<j})$ cannot be near scale on M, since that would imply that a'_1 was not maximal in the $\prec_M$-ordering by Lemma 5.1.11. $\square$

5.2 Definable n-Types

Previous Work

The concept of the “scale” of a nonuniquely realizable cut is closely related to the classification of definable types in o-minimal theories, as performed in [MS94]. We give some of their results here, and then refine them.

In this section, there are no ambient hypotheses on T besides its being an o-minimal expansion of a dense linear order without endpoints. Any further hypotheses will be stated when necessary.

First, we note that, for a 1-type, “definability” is exactly equivalent to being a noncut. However, in more variables this is no longer true. While a type in which each variable is a noncut over the preceding ones will be definable, there are other possibilities as well.

While [MS94]’s notation differs from ours, the following definition is equivalent to some of theirs.

Definition 5.2.1. Let $\mathrm{tp}(a/N)$ be *all out of scale on M* iff for every k and every N-definable k-ary function f, $f(M^k)$ is neither cofinal nor coinitial at a in N. Let $\mathrm{tp}(a/N)$ be *k-in-scale on M* iff for some k and some N-definable k-ary function f, $f(M^k)$ is cofinal and coinitial at a. Let $\mathrm{tp}(a/N)$ be *k-near scale on M* iff for some k and some N-definable k-ary function f, $f(M^k)$ is cofinal (or coinitial) at a.

Note that this definition does not make any requirement on $\mathrm{tp}(a/N)$ being a nonuniquely realizable cut. With this in hand, we can reproduce several results from [MS94]. Note that [MS94] use “Dedekind complete in” to mean “realizes no cuts over.”

Lemma 5.2.2. *([MS94], Lemma 2.5) Suppose that $p \in S_n(M)$, and $\langle c, d\rangle \models p$, with c a tuple and d a singleton. Let p_0 be the restriction of p to the first $n-1$ variables. If p_0 is definable and the type of d over Mc is not a cut, then p is definable.*

Lemma 5.2.3. *([MS94], Lemma 2.6) Suppose that (c, d) realizes an type p, where c is a tuple of length $n-1$, and d a singleton, the restriction p_0 of p to the first $n-1$ variables is definable, and that d realizes a cut over Mc, with* $\mathrm{tp}(d/Mc)$ *being all out of scale on M for some k. Then p is definable.*

Lemma 5.2.4. *([MS94], Lemma 2.7) Suppose that* (c, d) *realizes an type* p*, where* c *is a tuple of length* $n-1$*, and* d *a singleton, the restriction* p_0 *of* p *to the first* $n-1$ *variables is definable, and that* d *realizes a cut over* Mc*, with* $\mathrm{tp}(d/Mc)$ *being* k*-near scale on* M *for some* k*. Then* p *is definable.*

Lemma 5.2.5. *([MS94], Lemma 2.8) Suppose that* (c, d) *realizes* p*, with* c *a tuple of length* $n-1$ *and* d *a singleton, and that* d *realizes a cut over* Mc*. Suppose also that* $\mathrm{tp}(d/Mc)$ *is* k*-in scale on* M*. Then* p *is not definable and* $M(c, d)$ *realizes at least one cut over* M*.*

Using these and other lemmas, [MS94] obtain the following theorems.

Theorem 5.2.6. *([MS94], Theorem 2.1) Let* $p \in S_n(M)$*. Then* p *is definable iff for any* a *realizing* p*,* $M(a)$ *realizes no cuts over* M*.*

Theorem 5.2.7. *([MS94]) Let* p *be an* n*-type over a structure* M*. Let* $c = \langle c_1, \ldots, c_n \rangle$ *be a realization of* p*. Then* p *is definable iff for each* $i \leq n$*,* $\mathrm{tp}(c_i/Mc_{<i})$ *is a noncut, all out of scale on* M*, or* k*-near scale on* M*, for some* k*.*

Refinement

However, we can actually characterize definable types using concepts similar to our original "scale" definitions. We will be aided in this by a key result for the next several chapters. First, we tighten the connection between [MS94]'s definitions and ours.

Lemma 5.2.8. *Let* $M \prec N$*, suppose that* N *realizes no cuts in* M*, and let* $c \in N$ *be a tuple. Suppose that* $f(c, M^n)$ *is cofinal and/or coinitial at* d *in* N*, for* f *an* M*-definable function. Then there is a unary* N*-definable function,* g*, such that* $g(M)$ *is cofinal (coinitial) at* d *in* N *iff* $f(c, M^n)$ *is.*

Proof. We go by induction on n. The case $n = 1$ is trivial. Note that, since N realizes no cuts in M, by Theorem 5.2.6, c is definable over M. If $f(c, M^n)$ is cofinal and coinitial at d in N, then we are in the conditions of Lemma 5.2.5. But this implies that M is not Dedekind complete in $M(c, d)$. Thus, there must be some Mc-definable function, g, such that $g(d)$ is a cut in M – that is, since N realizes no cuts in M, $g^{-1}(M)$ is cofinal and coinitial at d in N, so we are done.

Now we show that if $f(c, M^n)$ is only cofinal (coinitial) at d in N, then we can find the appropriate g. WLOG, we assume $f(c, M^n)$ is cofinal at d in N. For notation, given a definable set, $D \subseteq \bar{M}^k$, let $D' = D \cap M^k$. Note that, by Lemma 5.2.4, we know that $\mathrm{tp}(c, d/M)$ is M-definable. Let $\hat{f}(-) = f(c, -)$. We can find an Mcd-definable cell, $C \subseteq \bar{M}^n$, such that $\hat{f}(C')$ is cofinal at d in N and $\hat{f}(C) < d$. Since $\mathrm{tp}(c, d/M)$ is M-definable, C can be taken to be M-definable. By induction on n, if, for any $a \in M$, $\hat{f}(a, C'_a)$ is cofinal at d in N, we are done. Thus, we may assume not. For each a, we have $\sup(\hat{f}(a, C'_a))$, which must be less than d. Since $\sup(\sup(\hat{f}(a, C'_a))) \geq \hat{f}(C')$, we know that $\sup(\hat{f}(a, C'_a))$ is cofinal at d in N, for $a \in M$. But then we are done, since we can take $g(x)$ to be $\hat{f}(x, C_x)$. ∎

With Lemma 5.2.8, we are able to replace "k-near scale" and "all out of scale" in Theorem 5.2.7 with "near scale" and "out of scale."

Theorem 5.2.9. *Let p be an n-type over M, and let $c \models p$. Then p is definable iff for $i \leq n$, $\operatorname{tp}(c_i/Mc_{<i})$ is a noncut, or 1-near scale or 1-out of scale on M.*

Proof. 1-near scale certainly implies k-near scale, and 1-out of scale implies all out of scale by Lemma 5.2.8. Similarly, all out of scale implies 1-out of scale, and k-near scale implies 1-near scale by Lemma 5.2.8. □

While heretofore we have only dealt with scale as it relates to nonuniquely realizable cuts, uniquely realizable cuts can also be analyzed with scales, and are very predictable, assuming that all noncuts over our base parameter set are interdefinable.

Lemma 5.2.10. *Let N be an extension of M that realizes no cuts, and let $\operatorname{tp}(d/N)$ be a uniquely realizable cut. Let all noncuts be interdefinable over M. Then $\operatorname{tp}(d/N)$ is all out of scale on M.*

Proof. Suppose not. Let f be N-definable such that $f(M^n)$ is cofinal (WLOG) at d in N. By Lemma 5.2.8, we may assume $n = 1$. We may restrict to an N-definable interval, I, such that $f(I \cap M)$ is cofinal at d in N, and f is monotone and continuous on I. WLOG, let f be increasing on I. Let $d' = f^{-1}(d)$, so $d' \in I$. Since d is a uniquely realizable cut over N, so is d', and M is cofinal at d' in N. Let $\epsilon \in N$ be a noncut above 0 over M. By hypothesis, there is some M-definable function, h, such that $h(\epsilon, d')$ is a noncut below d' over Md'. Since $\operatorname{tp}(d'/N)$ is uniquely realizable, there is some $e \in (h(\epsilon, d'), d') \cap N$, else the types of $h(\epsilon, d')$ and d' would be the same over N. Thus, there is some $a \in (e, d') \cap M$. But then $a \in (h(\epsilon, d'), d')$, so $h(\epsilon, d')$ is not a noncut below d' over Md', contradiction. □

Theorem 5.2.11. *Let p be an n-type over M, and let $c \models p$. Let all noncuts over M be interdefinable. Then p is definable iff for $i \leq n$, $\operatorname{tp}(c_i/Mc_{<i})$ is a noncut, a uniquely realizable cut (for $i > 1$), or near scale or out of scale on M.*

Proof. If p is definable, then $\operatorname{tp}(c_i/Mc_{<i})$ is a noncut or k-near scale or all out of scale on M. Note that, since $\operatorname{tp}(c_{<i}/M)$ is definable, $M(c_{<i})$ realizes no cuts over M, by Lemma 2.1.4. If $\operatorname{tp}(c_i/Mc_{<i})$ is k-near scale, then, since $\operatorname{tp}(c_{<i}/M)$ is definable, we are in the situation of Lemma 5.2.8, and thus $\operatorname{tp}(c_i/Mc_{<i})$ is near scale on M (it cannot be a uniquely realizable cut by Lemma 5.2.10). Since all out of scale implies uniquely realizable or out of scale on M, that finishes the forward direction.

For the reverse direction, we go by induction on n. Thus, we have $\operatorname{tp}(c_n/Mc_{<n})$ a noncut, a uniquely realizable cut, or near scale or out of scale on M. If $\operatorname{tp}(c_n/Mc_{<n})$ is a noncut, or near scale on M, then by Lemmas 5.2.2 and 5.2.4, p is definable. If $\operatorname{tp}(c_n/Mc_{<n})$ is uniquely realizable, then by Lemma 5.2.10, p is definable. If $\operatorname{tp}(c_n/Mc_{<n})$ is out of scale on M, then, since $\operatorname{tp}(c_{<n}/M)$ is definable by induction, $M(c_{<n})$ realizes no cuts over M, and thus, $\operatorname{tp}(c_n/Mc_{<n})$ is all out of scale, and hence, by Lemma 5.2.3, p is definable. □

Miscellaneous Results

Lemma 5.2.12. *Let $M \prec N$, let d be an element, and let b be a strictly $\prec_M$-maximal element over $N(d)$. Suppose that, for some N-definable function, $f(b, M(b))$ is cofinal (coinitial) at d in $N(b)$. Then there is some N-definable function, g, such that $g(M)$ is cofinal (coinitial) at d in N.*

Proof. First, note that if we can show that $f(M^2)$ is cofinal (coinitial) at d in N, then we will be done by Lemma 5.2.8, since N realizes no cuts over M (else b could not be strictly $\prec_M$-maximal). We may consider f on an N-definable cell, C, such that f is continuous and monotonic in each coordinate on C, and such that $f(C \cap M(b)^2)$ cofinal (coinitial) at d in $N(b)$. WLOG, we assume $f(C \cap M(b)^2)$ is cofinal.

Suppose $f(C \cap M^2)$ is not cofinal. Then we can find $a_1 \in N$ with $(a_1, d) \cap f(M^2) = \emptyset$. Since $f(b, M(b))$ is cofinal at d in $N(b)$, we can find α an M-definable function with $f(b, \alpha(b)) \in (a_1, d)$ (since every element of $M(b)$ is of the form $\alpha(b)$ for some such α).

If $\mathrm{tp}(b/M)$ is a noncut, WLOG above 0, then let $h(x) = f(x, \alpha(x))$. If h is decreasing, then $h^{-1}(a_1) > b$, but $h^{-1}(a_1)$ is necessarily a noncut above 0 over M, since any element of M between it and 0 would contradict choice of a_1. But this contradicts strict maximality of b. Thus, h is increasing. But then $h^{-1}(d) > b$. If $h^{-1}(d)$ is not a noncut near 0 over M, then there is some element of M between it and b, and that element contradicts choice of a_1, and if it is a noncut near 0, that contradicts the strict maximality of b. Thus, $\mathrm{tp}(b/M)$ cannot be a noncut, so we may assume $\mathrm{tp}(b/M)$ is a cut.

Then $\mathrm{tp}(f(b, \alpha(b))/N) \neq \mathrm{tp}(d/N)$, since otherwise, b would be interdefinable (over N) with an element in $\mathrm{tp}(d/N)$, and hence d would be interdefinable (over N) with an element in $\mathrm{tp}(b/M)$, which contradicts strict maximality of b, since then there would be an element of $N(d)$ that was a cut over M. Then we can choose $a_2 \in (f(b, \alpha(b)), d)$, $a_2 \in N$. Now consider the N-definable set $A = \{x_1 \in C \mid f(x_1, \alpha(x_1)) \in (a_1, a_2)\}$. Note that $b \in A$. Since $b \notin N$, we may assume that A contains an interval about b. Since $\mathrm{tp}(b/M)$ is a cut, N does not realize any elements of that type, and so this interval must contain points of M. Let b' be such a point. But then $g(b', \alpha(b') \in (a_1, a_2)$, contradiction. □

Lemma 5.2.13. *Let $M \prec N$, and let b be strictly $\prec_M$-maximal over N. Then $N(b)$ realizes no cuts over $M(b)$.*

Proof. Note that, since b is strictly $\prec_M$-maximal, N realizes no cuts over M. Suppose the conclusion fails, so let $f(b, e)$ be a cut over Mb, where f is N-definable, and e is a tuple from $N \setminus M$. We can choose f to minimize $k = \mathrm{lh}(e)$. Rearrange e as per Lemma 5.1.25 so that it is decreasing and there are no near scale or in scale cuts on M. We may also assume if $\mathrm{tp}(e_i/Me_{<i})$ is a noncut, it is a noncut above 0.

Case 1: $\mathrm{tp}(e_k/Me_{<k})$ **is a uniquely realizable cut**

Note that, since b is strictly $\prec_M$-maximal, $k \neq 1$, since otherwise $\mathrm{tp}(e_1/M)$ would be a cut, and therefore b would not be strictly $\prec_M$-maximal. Thus, $k > 1$, and $\mathrm{tp}(e_1/M)$ is a noncut above 0, implying that $\mathrm{tp}(e_1/Mb)$ is a noncut above 0.

Claim 5.2.14. $\mathrm{tp}(e_k/Mbe_{<k})$ *is a uniquely realizable cut.*

Proof. Let $q^{\pm}(x, y)$ be the $\emptyset$-definable functions taking a noncut, x, above 0 to a noncut above (below) y. Suppose the claim fails. Then either $\mathrm{tp}(e_k/Mbe_{<k})$ is a noncut, or a nonuniquely realizable cut. If it is a noncut, then there is some $Me_{<k}$-definable function, g, such that e_k is a noncut near $g(b)$ over $Mbe_{<k}$. Note that, since $\mathrm{tp}(e_k/Me_{<k})$ is a uniquely realizable cut, $g(b) \notin M(e_{<k})$, $\mathrm{tp}(g(b)/Me_{<k}) = \mathrm{tp}(e_k/Me_{<k})$, and g must be continuous

and monotone in a neighborhood of b whose image includes e_k. Then $g^{-1}(e_k)$ must have the same type as b over $Me_{<k}$, and, a fortiori, over M. If $\mathrm{tp}(b/M)$ is a cut, then this contradicts strict maximality, so $\mathrm{tp}(b/M)$ is a noncut. Since $g^{-1}(e_k)$ is a noncut near b over $Mbe_{<k}$, $b \in (q^-(e_1, g^{-1}(e_k)), q^+(e_1, g^{-1}(e_k)))$, and $\mathrm{tp}(b/M) = \mathrm{tp}(q^\pm(e_1, g^{-1}(e_k))/M)$ (since e_1 is a noncut above 0 over M). But then it is clear that b cannot be strictly $\prec_M$-maximal over N – one of $q^\pm(e_1, g^{-1}(e_k))$ would contradict that, using Lemma 5.1.4.

Thus, we may assume that $\mathrm{tp}(e_k/Mbe_{<k})$ is a nonuniquely realizable cut. This implies, using Lemma 2.2.17, that there is some b', a $Mbe_{<k}$-definable element, that is a noncut above 0 over $Me_{<k}$. But that implies that b is a noncut over $Me_{<k}$, contradicting Lemma 5.1.10. ☐

Let $q(e_1, f(b,e))$ be a noncut above $f(b,e)$ (over $Mf(b,e)$) definable from Me_1. Since $\mathrm{tp}(e_k/Me_{<k}b)$ is a uniquely realizable cut, by Lemma 2.2.15 we may choose $e'_k \in M(be_{<k})$ such that $f(b, e_{<k}, e'_k) \in (f(b,e), q(e_1, f(b,e)))$, but then $\mathrm{tp}(f(b, e_{<k}, e'_k)/Mb) = \mathrm{tp}(f(b,e)/Mb)$, contradicting minimality of k.

Case 2: $\mathrm{tp}(e_k/Me_{<k})$ **is a noncut**

Claim 5.2.15. $\mathrm{tp}(e_k/Mbe_{<k})$ *is a noncut above* 0*.*

Proof. Suppose not. Then there is some $Mbe_{<k}$-definable element, $\beta(b)$, with β an $Me_{<k}$-definable function, such that $0 < \beta(b) < e_k$. But then b is a noncut over $Me_{<k}$, contradicting Lemma 5.1.10. ☐

Since $\mathrm{tp}(e_k/Mbe_{<k})$ is a noncut above 0, $f(b,e)$ is a noncut over $Mbe_{<k}$. Since $f(b,e)$ is a cut over Mb, there must be an element of $M(be_{<k})$ that is a cut over Mb, contradicting minimality of k.

Case 3: $\mathrm{tp}(e_k/Me_{<k})$ **is a nonuniquely realizable cut**

We know that $\mathrm{tp}(e_k/Me_{<k})$ is out of scale on M, by choice of e. Since N realizes no cuts over M, we know that $f_b^{-1}(M'(b))$ is cofinal and coinitial at e_k in $M(e_{<k}b)$. By Lemma 5.2.12, with $N = M(e_{<k})$ and $d = e_k$, we know that there is a unary $Me_{<k}$-definable function, g, such that $g(M)$ is cofinal and coinitial at e_k in $M(e_{<k})$. But then $g^{-1}(e_k)$ must realize a cut over M, contradiction. ☐

Lemma 5.2.16. *Let $M \prec N$, and let b be a decreasing tuple of length n, with $N(i) \leq 1$ for all $i \leq n$, and b_n strictly $\prec_M$-maximal over N. Let d be an element, and let $f(x,y)$ be N-definable, with x a tuple of length n and y a singleton, and $f(b, M(b))$ cofinal (coinitial) at d in $N(b)$. Then there is some N-definable $g(y)$, with y a singleton, such that $g(M)$ is cofinal (coinitial) at d in N.*

Proof. We show that such a g exists by induction on n, simultaneously for all such M, N, b, d, and f. The case $n = 1$ is trivial.

Suppose we know the result for n, and we are trying to show it for $n+1$. Suppose we have such M, N, b, d, and f satisfying the premise of the lemma. Let $M' = M(b_1)$ and $N' = N(b_1)$. Then I claim M', N', $b_{>1}$, d, and $f(b_1, -)$ satisfy the premise of the lemma.

The only condition to check is that b_{n+1} is strictly $\prec_{M'}$-maximal over N'. Suppose not, so let $c \in N' \setminus M'$, such that c is a cut over Mb_1b_{n+1}.

Claim 5.2.17. *c is a noncut over Mb_1.*

Proof. Since b_1 is $\prec_M$-maximal over b, and b_n is strictly $\prec_M$-maximal over N, b_1 is strictly $\prec_M$-maximal over N, and thus, since N realizes no cuts over M, $N(b_1)$ realizes no cuts over Mb_1. □

$\mathrm{tp}(b_{n+1}/Mb_1)$ is a cut, by Lemma 5.1.23. But then, since c is a cut over Mb_1b_{n+1}, b_{n+1} is a noncut over Mb_1, which this is impossible. Thus, we are in the premise of the lemma and, by induction, know that there is some N'-definable function, g, such that $g(M')$ is cofinal (coinitial) at d in N'. But then, by Lemma 5.2.12, we know that there is an N-definable function, h, such that $h(M)$ is cofinal (coinitial) at d in N. □

Lemma 5.2.18. *Let $M \prec N$, let d be an element, and let b be a decreasing tuple of length n, with $N(i) \leq 1$ for all $i \leq n$, and b_n strictly $\prec_M$-maximal over $N(d)$. Then N is dense in $N(b)$ in an interval around d.*

Proof. Lemma 5.1.10 gives the result for $n = 1$. Assume for a contradiction that $\mathrm{tp}(b_n/Nb_{<n})$ is a noncut, for some $n > 1$. Then there is some $Mb_{<n}$-definable function, $f(x)$, with x a tuple, such that $\mathrm{tp}(f(c)/Mb_{<n}) = \mathrm{tp}(b_n/Mb_{<n})$ for some tuple $c \in N$. But we know that, since N realizes only noncuts over $Mb_{<n}$, (as $M(b_{<n})$ realizes only cuts over Mb_1 and N realizes only noncuts over Mb_1), $\mathrm{tp}(f(c)/Mb_{<n})$ is a noncut, and $\mathrm{tp}(b_n/Mb_{<n})$ is a cut, contradiction. □

Lemma 5.2.19. *Let p be a decreasing type in n variables over a set A. Let f be an $Ac_{<n}$-definable function. Then $f(Ac_{<N(n)})$ is cofinal (coinitial) at c_n in $\mathrm{dcl}(Ac_{<n})$ iff there is some $Ac_{<n}$-definable g such that $g(A)$ is cofinal (coinitial) at c_n in $\mathrm{dcl}(Ac_{<n})$.*

Proof. The reverse direction is trivial. For the forward direction, we go by induction on n. Note that if $N(n) = 1$ or $N(n) = 0$, the result is trivial, so we may assume not. Let $k = N(N(n) - 1)$. Now, for each c_j, $N(n) \leq j \leq n$, c_j is a noncut over $c_{<N(n)}$. Replace each c_j by c'_j, a noncut above 0 definable from $c_{<N(n)}c_j$. Let $c' = \langle c'_{N(n)}, \ldots, c'_{n-1}\rangle$. Then c' is a decreasing sequence over $Ac_{<N(n)}$, and actually over $Ac_{<k}$, by Lemma 5.1.5.

Then, with $M = \mathrm{Pr}(Ac_{<k})$, $N = M(c'_{N(n)}, \ldots, c'_{n-1})$, $b = \langle c_k, \ldots, c_{N(n)-1}\rangle$, and $d = c'_n$, by Lemma 5.2.16, there is an N-definable function, g, such that $g(M)$ is cofinal (coinitial) in N at c'_n. In other words, there is an $Ac_{<k}c'_{\geq N(n)}$-definable function, g, such that $g(\mathrm{dcl}(Ac_{<k}))$ is cofinal (coinitial) in $\mathrm{dcl}(Ac_{<k}c'_{\geq N(n)})$ at c'_n. By induction, this implies that there is an $Ac_{<k}c'_{\geq N(n)}$-definable function, h, such that $h(\mathrm{dcl}(A))$ is cofinal (coinitial) in $\mathrm{dcl}(Ac_{<k}c'_{\geq N(n)})$. But then h is $Ac_{<n}$-definable, and, since by Lemma 5.2.18, N is dense in $N(b)$, h is cofinal (coinitial) in $N(b)$, i.e., in $\mathrm{dcl}(Ac)$. □

Corollary 5.2.20. *Let A be a set, and let c be a decreasing sequence over A, of length n. Suppose $\mathrm{tp}(c_n/Ac_{<n})$ is a nonuniquely realizable cut. Then $\mathrm{tp}(c_n/Ac_{<n})$ is in scale (near scale) on A iff it is in scale (near scale) on $Ac_{<N(n)}$.*

Proof. Apply Lemma 5.2.19 to the function witnessing in scale or near scale. □

5.3 Infinite Decreasing Extensions

Introduction

Lemma 5.1.20 tells us that any n-type can be reordered to be decreasing. The question remains, though, whether any type, in any number of variables, can be reordered to be decreasing. First we give a definition of decreasing for infinite sequences.

Definition 5.3.1. Let I be any order type. We denote "J is a proper initial segment of I" by $J \sqsubset I$.

Definition 5.3.2. Let $c = \langle c_i \rangle_{i \in I}$ be a sequence, with order type I. Say c is *decreasing* if, for any $J \sqsubset I$, there is $S(J) \sqsupset J$, $S(J) \sqsubset I$, such that for any $j \in S(J) \setminus J$ and $k > j$, $c_j \gtrsim_{c_J} c_k$. We will denote $\gtrsim_{c_J}$ by $\gtrsim_J$. Note that this definition extends the definition in the finite case, where $S(J)$ is the initial segment extending J isomorphic to $J + 1$. Decreasing types are defined analogously to the finite case.

While it is easy to see that a finite sequence can be reordered so that it is decreasing, it is not so simple to reorder an infinite sequence.

Example 5.3.3. Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$, and let $\langle \epsilon_i \rangle_{i \in \mathbb{Z}}$ be a sequence of noncuts above 0, such that $\mathrm{tp}(\epsilon_i / M\epsilon_{<i})$ is a noncut above 0. Let α be the cut approximated by $\sum_{0 \le i} \prod_{0 \ge j \ge -i} \epsilon_j$ (meaning that α is greater than every finite partial sum of this expression, but less than any element of M that is greater than every finite partial sum). Let $\beta_k = \epsilon_k + \sum_{k \le i} \prod_{-k \ge j \ge -i} \epsilon_j$ for $k > 0$. Let $C = \{\epsilon_i\}_{i \in \mathbb{Z}} \alpha \{\beta_k\}_{k \in \mathbb{N}}$. Then, while C can be ordered to be decreasing, it requires some care, in that the β_k's must come before the ϵ_i's.

This example shows that the method used in Lemma 5.1.20 can fail for infinite extensions – namely, care is required in the order that elements are selected. If, for example, we constructed the sequence by inserting the ϵ_i's first, there would be no way to insert the β_i's to keep the sequence decreasing. I do not know if the analogue of Lemma 5.1.20 is actually true for infinite extensions.

However, if we assume that the set is algebraically closed – as it will be if it is an elementary extension of our base – then we can obtain the expected result, which is analogous to Lemma 5.1.25.

For the rest of this section, we will fix a base model, M, and an elementary extension, N. For ease of notation, we expand our language by constants for each element of M, so that $\mathrm{tp}(a)$ is equivalent to $\mathrm{tp}(a/M)$, and similarly $\emptyset$-definable is equivalent to M-definable.

Convex Sets

Definition 5.3.4. Let $S \subseteq N$. We form the *T-convex closure* of S by taking the convex closure of $\mathrm{dcl}(S) \cap N^+$, denoted $\mathrm{tcl}(S)$.

Let $\mathcal{S} = \{\mathrm{tcl}(S) \mid S \subseteq N^+\}$.

Lemma 5.3.5. *$\mathcal{S}$ is totally ordered by inclusion.*

Proof. Suppose not, so let S_1, S_2 be such that $\operatorname{tcl}(S_1) \setminus \operatorname{tcl}(S_2)$ and $\operatorname{tcl}(S_2) \setminus \operatorname{tcl}(S_1)$ are both non-empty. Since both are convex, it must be the case that, WLOG, there is $s_1 \in \operatorname{tcl}(S_1)$, with $(0, s_1] \cap \operatorname{tcl}(S_2) = \emptyset$, and $s_2 \in \operatorname{tcl}(S_2)$, with $[s_2, \infty) \cap \operatorname{tcl}(S_1) = \emptyset$. We may assume that $s_1 \in \operatorname{dcl}(S_1)$, $s_2 \in \operatorname{dcl}(S_2)$ (otherwise, since they are in the convex closures, we may replace them with more "extreme" elements). But the map, q, sending the noncut near ∞ to the noncut above 0 shows that either $q(s_2) \in (0, s_1)$, so $\operatorname{dcl}(S_2) \cap (0, s_1) \neq \emptyset$, contradiction, or that, similarly, $\operatorname{dcl}(S_1) \cap (0, s_2) \neq \emptyset$, also contradiction. □

Let $\mathcal{S}$ be given by $\langle U_i \rangle_{i \in I_S}$, for I_S some ordered index set. For each $i \in I_S$, let c_i be any element of U_i which is less than $\bigcup_{j<i} U_j$, if such element exists. Let the ordered index set of the c_i's be given by $I_0 \subseteq I_S$.

Lemma 5.3.6. *If $i > J$, for $J \sqsubset I^0$, then $\operatorname{tp}(c_i/c_J)$ is a noncut above* 0.

Proof. Suppose not, so for some initial segment J and $i > J$, we have $0 < f(e) < c_i$, with e a tuple from c_J and f an $\emptyset$-definable function. Choose J and i to minimize $\operatorname{lh}(e)$. Note that $\operatorname{tp}(c_i)$ is a noncut above 0, since $c_i < M^+$, so $\operatorname{lh}(e) \geq 1$. Let $e = \langle c_{i_1}, \ldots, c_{i_k} \rangle$, with $i_1 < \ldots < i_k$, and let $e_j = c_{i_j}$, for $j = 1, \ldots, k$. Since e has minimal length, $\operatorname{tp}(e_j/e_{<j})$ is necessarily a noncut above 0 for $1 \leq j \leq k$. But by Theorem 3.2.2, this means that $f(e) = f(e_1, \ldots, e_{k-1}, e_k) \geq g(e_k) > 0$, for some $\emptyset$-definable g. Thus, we may take e to be a singleton, c_j, and so $f(c_j) < c_i$. By the same argument as above, we know that $\operatorname{tp}(c_j)$ is a noncut above 0. Thus, since $c_j \in U_j$, there must be $d \in \operatorname{dcl}(S_j)$ with $d \in (0, c_j]$. But then $f(d) \leq f(c_j) < c_i$, so $c_i \in \operatorname{tcl}(S_j) = U_j$, contradicting choice of c_i. □

Lemma 5.3.7. *c^0 is decreasing. In fact, $i < j$ implies $c_j \precsim_J c_i$, for any $J \sqsubset I^0$.*

Proof. We need only show the claim for i, j, J with $i > J$. By the previous claim, both c_i and c_j are noncuts above 0 over c_J. Since $c_j \notin U_i$, $c_j < c_i$. Given any $\emptyset$-definable function, f, and tuple e from c_J, if $f(e, -)$ is increasing in a neighborhood above 0, mapping it to a neighborhood above 0, then $0 < f(e, c_j) < f(e, c_i)$, showing that $c_j \precsim_J c_i$. □

Lemma 5.3.8. *For $i \in I_S$, if c_i exists, then $U_i = \operatorname{tcl}(c_i)$.*

Proof. Note first that $\operatorname{tcl}(c_i) \in \mathcal{S}$, and $\operatorname{tcl}(c_i) \neq U_j$ for $j < i$, since c_i was chosen not in U_j for $j < i$. Thus, $\operatorname{tcl}(c_i) \supseteq U_i$. It remains to show that $U_i \supseteq \operatorname{tcl}(c_i)$, which we do by proving that an arbitrary $d \in \operatorname{tcl}(c_i)$ is in U_i. Given $d \in \operatorname{tcl}(c_i)$, we know $(0, d] \cap \operatorname{dcl}(c_i) \neq \emptyset$. We may assume d is a noncut above 0 or near ∞, since otherwise our goal is trivial. WLOG, assume it is above 0. Let f be an $\emptyset$-definable function with $f(c_i) \in (0, d]$. Thus, f is increasing in a neighborhood of 0. Since $c_i \in U_i$, there is some $e \in \operatorname{dcl}(S_i)$ with $e \in (0, c_i]$. But then $f(e) \leq f(c_i) \leq d$, showing that $d \in U_i$. □

Lemma 5.3.9. *For $i \in I^0$, $\operatorname{tcl}(c_i) = \operatorname{tcl}(c_{\leq i})$.*

Proof. This is an easy consequence of Theorem 3.2.2. For it not to be the case, $\operatorname{dcl}(c_{\leq i})$ would have to realize a noncut above 0 over c_i, through some $f(c_i)$, with f a $c_{<i}$-definable function (note that, since $\operatorname{tp}(c_i/c_{<i})$ is a noncut above 0 by Lemma 5.3.6, c_i is a necessary argument to f). But Theorem 3.2.2 gives us an $\emptyset$-definable function, g, such that $0 < g(c_i) \leq f(c_i)$, contradiction. □

Lemma 5.3.10. *For any initial segment, $J \sqsubseteq I_S$, $\bigcup_{i\in J} U_j \in \mathcal{S}$.*

Proof. We show that $\bigcup_{i\in J} U_j = \mathrm{tcl}(c_J)$. Clearly, $\bigcup_{i\in J} U_j \subseteq \mathrm{tcl}(c_J)$, by Lemma 5.3.8 – indices i such that c_i exists must be cofinal in J. In the other direction, let $d \in \mathrm{tcl}(c_J)$, with d a noncut above 0, WLOG. Then $(0, d] \cap \mathrm{dcl}(c_J) \neq \emptyset$, so for some *emptyset*-definable function, f, $f(c_{i_1}, \ldots, c_{i_k}) \in (0, d]$, with $i_1 < \ldots < i_k$. But then $d \in \mathrm{tcl}(c_{\leq i_k}) = U_{i_k}$, so we are done. □

Claim 5.3.11. *Let $A \subseteq N$ be a set. If b is strictly $\prec_A$-maximal with respect to the set $c^0 \setminus \mathrm{dcl}(A)$, then b is a cut over A.*

Proof. Suppose that b were a noncut over A. Replacing b by a Ab-definable element, we can assume that b is a noncut above 0 over A. Let $U_i = \mathrm{tcl}(A)$, and $U_j = \mathrm{tcl}(Ab)$, with $i < j \in I_S$. We first show that $U_j = \mathrm{tcl}(b)$. Clearly $\mathrm{tcl}(b) \subseteq U_j$. As well, $\mathrm{tcl}(b) \supseteq \mathrm{tcl}(A)$, since b and the image of b under the function sending the noncut above 0 to the noncut near ∞ bounds $\mathrm{tcl}(A)$. Moreover, if $d \in U_j \setminus \mathrm{tcl}(A)$, WLOG a noncut above 0 over MA, then, for some A-definable f, $f(b) \in (0, d]$, but then, by Theorem 3.2.2, there is some $\emptyset$-definable g with $0 < g(b) \leq f(b)$, so $d \in \mathrm{tcl}(b)$.

We know there is some k, $i < k \leq j$, such that c_k exists. We show that $b \precsim_A c_k$, yielding a contradiction. Since $\mathrm{tcl}(c_k) \subseteq \mathrm{tcl}(b)$, there must be some $\emptyset$-definable function, f, such that $0 < f(b) \leq c_k$, which is enough, by Lemma 5.1.4. □

Before we continue to our main result, it is worth examining a related concept in the case that T expands the theory of an ordered field.

The Maximal Valuation

We first introduce the "maximal valuation" on N.

Definition 5.3.12. . Consider $N^\times$, the multiplicative group of N. Define $a \sim_v b$ iff $ca \leq b \leq da$, for $c, d \in M$. Let $\Gamma = N^\times / \sim_v$, and write Γ additively. Let $v : N^\times \to \Gamma \cup \{\infty\}$ be defined by sending elements to their images under quotienting, and sending 0 to ∞.

Lemma 5.3.13. *v is a valuation, with Γ the value group.*

Proof. Note that, for any $g \in \Gamma$, $|v^{-1}(g)|$ is connected, by definition of $\sim_v$. We can order Γ by saying $a < b \iff \exists c \in |v^{-1}(b)|((0, c) \cap v^{-1}(a) = \emptyset)$. It is clear that this is a partial order, and the fact that it is total follows easily since $v^{-1}(x)$ is always connected.

To show that v is a valuation, it remains to verify $v(xy) = v(x) + v(y)$, and $v(x + y) \geq \min\{v(x), v(y)\}$. The first is trivial. For the second, note that, if $v(x) = v(y)$, then for some $c, d \in M$, $cx \leq y \leq dx$, and so $(c + 1)x \leq x + y \leq (d + 1)x$. The conclusion follows easily from this. □

Lemma 5.3.14. *If $v(a) = v(b) \neq 0$, then $\mathrm{tp}(a) = \mathrm{tp}(b)$.*

Proof. Since order type determines type, if $\mathrm{tp}(a) \neq \mathrm{tp}(b)$, then there is some $c \in M$, $a < c < b$ (WLOG). But then $v(a) > 0$, $v(b) < 0$. □

We can now link subgroups of the value group with T-convex closures.

Lemma 5.3.15. *If $S \subseteq N^+$, S non-empty, then $\tilde{S} = v(\mathrm{tcl}(S))$ is a group.*

Proof. Let $x \in \tilde{S}$. Then for some $a \in \mathrm{tcl}(S)$, $v(a) = x$. If $a \in \mathrm{dcl}(S)$, then $1/a \in \mathrm{dcl}(S)$, so $v(1/a) = -x \in \tilde{S}$. If $a \notin \mathrm{dcl}(S)$, then, for some $s_1, s_2 \in \mathrm{dcl}(S)$, $s_1 < a < s_2$, so $1/s_2 < 1/a < 1/s_1$, so again $1/a \in \mathrm{tcl}(S)$. This shows the presence of inverses. Convexity of S shows the presence of 0. If $x, y \in \tilde{S}$, we must show $x + y \in \tilde{S}$. Let $a, b \in \mathrm{tcl}(S)$ be such that $v(a) = x$, $v(b) = y$. By definition of the valuation, $x + y = v(ab)$. Again, if $a, b \in \mathrm{dcl}(S)$, then $ab \in \mathrm{tcl}(S)$, and if one or both are not in $\mathrm{dcl}(S)$, we may bound them by elements that are, and then bound ab by the products of those elements, again yielding $ab \in \mathrm{tcl}(S)$. ☐

Main Result

Theorem 5.3.16. *Let N be an elementary extension of M. Then we can write $N \setminus M$ as a sequence, $c = \langle c_i \rangle_{i \in I}$, for some ordered set I, such that c is decreasing over M.*

Proof. Let $|N \setminus M| = \kappa$. We begin with a "skeleton" $c^0 = \langle c_i \rangle_{i \in I^0}$, with the c_i's from above in the discussion on $\mathcal{S}$.

We now have c^0, a decreasing sequence in N. We must expand it, keeping it decreasing, to include all elements of $N \setminus M$. First, we expand it to c' such that $\mathrm{dcl}(c') = N$. It will then be trivial to expand c' to c such that c is an ordering of $N \setminus M$.

Let $|N \setminus (M \cup c^0)| = \kappa$. We construct c' by induction in $< \kappa^+$-many stages, ordinal-indexed. At stage α, we have c^α, a decreasing sequence indexed by ordered set I^α, with $I^\beta \subseteq I^\alpha$ if $\beta \leq \alpha$. We denote c^α by d for ease of notation.

We have the following three induction assumptions.

(I1) d is decreasing.

(I2) For any $i \in I^\alpha$, if there is a maximal $J \sqsubset I_\alpha$ such that $\mathrm{tp}(d_i/d_J)$ is a noncut, then it is a noncut above 0.

(I3) For every $K \sqsubset I^\alpha$, if $a \in \mathrm{dcl}(d_K) \setminus d_K$, then a is not strictly J-maximal for any initial segment $J < K$ unless $a \in \mathrm{dcl}(d_J)$.

Preparation

We first prove some useful claims.

Claim 5.3.17. *For any $b \in N$, there is a shortest initial segment of I^α, $J(b)$, such that b is strictly $J(b)$-maximal.*

Proof. Since b is strictly I^α-maximal, we can let $J(b) = \bigcap\{K \sqsubseteq I^\alpha \mid b \text{ is strictly } K\text{-maximal}\}$. Suppose that b is not strictly $J(b)$-maximal. Then we can find J' with $\langle J(b), J' \rangle$ an initial segment of $I\alpha$, such that, for $j \in J'$, $b \precsim_{J(b)} d_j$. Suppose that, for some $j \in J'$, d_j is a noncut over $d_{<j}$. Then, by (I2), it is a noncut near 0. Thus, if $b \succ_{<j} d_j$, then $b \succ_J d_j$, contradiction. Therefore, every element of $d_{J'}$ is a cut over its predecessors. But then, for $j \in J'$, b cannot be $<j$-maximal, because d_j is a cut over $d_{<j}$. Contradiction, and so b is strictly J-maximal. ☐

The following claims will usually be used with $J = J(b)$. However, note that they do not depend on that, and we will use them occasionally with J not necessarily $J(b)$.

Claim 5.3.18. *If b is strictly J-maximal, b is a cut over d_J.*

Proof. Apply Claim 5.3.11 to d_J. $\square$

Claim 5.3.19. *If b is strictly J-maximal, $J \sqsubset K$ and d_K realizes no cuts over d_J, then* $\mathrm{dcl}(d_K)$ *realizes no cuts over d_K.*

Proof. Trivial, using (I3). $\square$

Claim 5.3.20. *If b is strictly J-maximal, then* $\mathrm{tp}(b/d_K)$ *is a cut for any initial segment K extending J.*

Proof. By Claim 5.3.19, we know that $\mathrm{dcl}(d_K)$ realizes no cuts over d_J, since the fact that b is strictly J-maximal implies that d_K realizes no cuts over d_J. If $\mathrm{tp}(b/d_K)$ were not a cut, $\mathrm{dcl}(d_K)$ would have to realize an element with the same type as b over d_J, but then $\mathrm{dcl}(d_K)$ would realize a cut, contradiction. $\square$

Claim 5.3.21. *Let b be strictly J-maximal. For $j > K$, K an initial segment extending J,* $\mathrm{tp}(d_j/d_K)$ *is a cut iff* $\mathrm{tp}(d_j/d_Kb)$ *is a cut.*

Proof. Suppose not. First, assume that $\mathrm{tp}(d_j/d_K)$ is a cut, but $\mathrm{tp}(d_j/d_Kb)$ is a noncut. Then there is some d_K-definable function, f, such that $f(b)$ is a noncut near d_j over d_K. But then $f^{-1}(d_j)$ is a noncut near b over d_K, and hence a fortiori over d_J. But then $f^{-1}(d_j)$ is strictly J-maximal, implying that $f^{-1}(d_j) \in d_K$ by (I3), and so b is not strictly J-maximal, contradiction.

Now, assume that $\mathrm{tp}(d_j/d_K)$ is a noncut. Then $\mathrm{dcl}(d_jd_K)$ realizes no cuts over d_K. Since b is d_K-maximal over $\mathrm{dcl}(d_Kd_j) \setminus \mathrm{dcl}(d_K)$ by Claim 5.3.20, we can apply Lemma 5.2.13, showing that $\mathrm{tp}(d_j/d_Kb)$ is a noncut. $\square$

Claim 5.3.22. *Let b be strictly J-maximal. Let $B_J(b) = \mathrm{dcl}(d_Jb) \setminus \mathrm{dcl}(d_J)$. Let $a \in \mathrm{dcl}(db)$. If a is strictly J'-maximal for $J' < J$, then there is some $b' \in B_J(b)$ such that b' is strictly J'-maximal and a is a noncut near b' over $d_{J'}$ – in fact, over d_J. Moreover, a is never strictly maximal over $d_{>J'}$ with respect to $\prec_{d_{J'}b}$ for $J' > J$.*

Proof. First we consider the case where $J' < J$. Since b is strictly J-maximal, no element of $d_{>J}$ realizes a cut over d_J, and, in fact, no element of $\mathrm{dcl}(d_{>J})$ realizes a cut over d_J, by I3. As well, by Claim 5.3.18, $\mathrm{tp}(b/d_J)$ is a cut. Thus, we can apply Lemma 5.2.13, so $\mathrm{dcl}(db)$ realizes no cuts over $\mathrm{dcl}(d_Jb)$. Let $a \in \mathrm{dcl}(db)$ be strictly J'-maximal. Then $\mathrm{tp}(a/d_{J'})$ is a cut by Claim 5.3.11, and hence, by Claim 5.3.20, $\mathrm{tp}(a/d_J)$ is a cut. Since $\mathrm{tp}(a/d_Jb)$ is a noncut, we can find $b' \in B_J(b)$ such that a is a noncut over d_J near b'.

By Claim 5.3.11, if $a \in \mathrm{dcl}(db)$ is strictly J'-maximal, then it must be a cut over $d_{J'}$. Moreover, since d_J realizes no cuts over $d_{J'}$, $\mathrm{tp}(a/d_J)$ is still a cut. By Claim 5.3.21, this implies $\mathrm{tp}(a/d_Jb)$ is a cut, contradiction.

If $J' > J$, suppose a is strictly $d_{J'}b$-maximal over $d_{>J'}$. By Claim 5.3.11, $\mathrm{tp}(a/d_{J'}b)$ is a cut. Since a is strictly $d_{J'}b$-maximal, that implies that $\mathrm{tp}(d_j/d_{J'}b)$ is a noncut, for $j > J'$.

By Claim 5.3.21, that implies that $\mathrm{tp}(d_j/d_{J'})$ is a noncut, for $j > J'$, and thus, by I3, $\mathrm{dcl}(d)$ realizes no cuts over $\mathrm{dcl}(d_{J'})$. But then, since by Claim 5.3.20 $\mathrm{tp}(b/d_{J'})$ is a cut, and hence strictly J'-maximal, Lemma 5.2.13 implies $\mathrm{tp}(a/d_{J'}b)$ is not a cut, contradiction. ☐

Claim 5.3.23. *If b is strictly $J(b)$-maximal, then $d_{J(b)} \frown \{b\}$ is decreasing.*

Proof. Suppose not. Let K be an initial segment of $J(b)$ witnessing the failure. Then, for any J' extending K, there is $j \in J' \setminus K$ such that $d_j \prec_K b$. But that means that b is strictly K-maximal, contradicting the definition of $J(b)$. ☐

Successor stage – constructing $c^{\alpha+1}$

Fix $b \in N \setminus \mathrm{dcl}(d)$. Fix J to be the shortest initial segment guaranteed by Claim 5.3.17. If there is a maximal $K \sqsubset J$ such that b is a noncut over d_K, we may replace b by a bd_K-definable element that is a noncut above 0 over d_K. Let $B = \mathrm{dcl}(d_J b) \setminus \mathrm{dcl}(d_J)$. Let $J_0 = \bigcap\{K \sqsubseteq I^\alpha \mid \exists b' \in B(b' \text{ is } K\text{-maximal})\}$.

Case 1: There is $b' \in B$ such that b' is strictly J_0-maximal

In this case, we replace b by b' and insert b after d_{J_0}. Let $c^{\alpha+1} = \langle d_{J_0} b, d_{>J_0}\rangle$. Let $d' = c^{\alpha+1}$. Let I' be the index set of d'. Let γ be the index of b, so $I' \setminus \{\gamma\} = I^\alpha$. In general, for $K' \sqsubset I'$, let $K = K \setminus \{\gamma\}$, and for $K \sqsubset I^\alpha$, let K' be the initial segment of I' formed by inserting γ at the appropriate point (if $J_0 \sqsubset K$).

Claim 5.3.24. *d' is decreasing.*

Proof. Suppose not. By Claim 5.3.23, and since b is strictly J_0-maximal, the counterexample must come for some J' extending the segment $J_0 \frown \{\gamma\}$. Then, for any K' extending J', there is some $j \in K'$, $k > j$, such that $d'_k \succ_{J'} d'_j$. We can take K' to be $S(J)'$. Thus, there is some element of $\mathrm{dcl}(d_J d_j b)$ that is a noncut near 0 over $\mathrm{dcl}(d_J d_k b)$, but $\mathrm{dcl}(d_J d_j)$ realizes no noncuts over $\mathrm{dcl}(d_J d_k)$. Thus, there is some d_J-definable function, f, such that d_j is a noncut near $f(b)$ over $d_J d_k b$. But then we can pull d_j back, d_J-definably, to get $f^{-1}(d_j)$, a noncut near b over d_J. Since b is strictly J_0-maximal, so is $f^{-1}(d_j)$, so $f^{-1}(d_j) \in d$ by (I3), violating the fact that b is J_0-maximal. ☐

Claim 5.3.25. *d' satisfies I2.*

Proof. By Claim 5.3.21, if $\mathrm{tp}(d'_j/d'_{K'})$ is a noncut, then $\mathrm{tp}(d_j/d_K)$ is a noncut, for any K extending J_0. As well, since b is a cut over d_K by Claim 5.3.20, it is not possible for $\mathrm{tp}(d'_j/d_K)$ to be a noncut near 0, while $\mathrm{tp}(d'_j/d'_{K'})$ is a noncut not near 0. So we are done. ☐

Claim 5.3.26. *d' satisfies I3.*

Proof. We assumed that b does not violate I3, so any failure must come from K' extending $J_0 \frown \{\gamma\}$. Suppose there is a failure at K', so let $a \in \mathrm{dcl}(d'_{K'}) \setminus d'_{K'}$, with a strictly J'-maximal for some $J' \sqsubset K'$. Since a is strictly J'-maximal, we know that $d_{>J}$ $(= d'_{>J'})$ does not realize any cuts over $d_J b$. By Claim 5.3.21 and (I3), this means that $\mathrm{dcl}(d)$ realized no cuts over d_J. But then by Lemma 5.2.13, $\mathrm{dcl}(db)$ realizes no new cuts over bd_J, which contradicts the existence of a. ☐

Case 2: Case 1 fails

In this case, we will need to adjoin all elements of B that are J'-maximal for some $J' \sqsubset J$. We do this in stages, ordinal-indexed. We have $d^0 = d$, indexed by $J^0 = I^\alpha$, $C^0 = J$, and $B^0 = B$.

At each successor stage $i+1$, choose $b' \in B^i$ such that b' is strictly $d^i_{J'}$-maximal (over $d^i_{>J'}$) and $b' \notin \mathrm{dcl}(d^i_{J'})$, for J' some initial segment of J^i and $J' \sqsubset C^i$. Note that, if b' exists and there is some K such that b' is a noncut over d^i_K, then we can assume that b' is a noncut above 0 over d^i_K. If we can choose K maximal, do so, otherwise, choose arbitrary K with $J_0 \sqsubset K$, if such K exists. If this b' exists, then insert b' into d^i after the first such segment that it is maximal over. Let d^{i+1} be this new sequence. Let $B^{i+1} = \mathrm{dcl}(d^i_{J'} b')$, let C^i be the K we chose, and let J^{i+1} be the index set of d^{i+1}. Take unions/intersections at limits. This construction must halt in fewer than $|B|^+$ stages.

Claim 5.3.27. *Let $\beta \in d^{i_1}$, for some stage i_1. Let $i_2 > i_1$. For $K \sqsubset J^{i_2}$, let $K' = K \cap J^{i_1}$. Then β is a noncut near $e \in \mathrm{dcl}(d^{i_2}_K)$ over $d^{i_2}_K$ iff β is a noncut near e over $d^{i_1}_{K'}$ (in particular, $e \in \mathrm{dcl}(d^{i_1}_{K'})$).*

Proof. If $d^{i_1}_{K'} = d^{i_2}_K$, then the conclusion is trivial, so assume not. Then we can choose the first element that was added in $d^{i_2}_K \setminus d^{i_1}_{K'}$, b'. Then every element added is algebraic over $b' d^{i_1}_{K'}$.

First, suppose β is a noncut near e over $d^{i_1}_{K'}$. If β is not a noncut near e over $d^{i_2}_K$, then there is some $e' \in \mathrm{dcl}(d^{i_2}_K)$ between e and β (with "between" having the obvious interpretation if $e = \pm\infty$). We may assume that $e' = f(b')$, where f is a $d^{i_1}_{K'}$-definable function. Since $e' \notin \mathrm{dcl}(d^{i_1}_{K'})$, we know that e' and β must have the same (noncut) type over $d^{i_1}_{K'}$. But b' realizes a cut over $d^{i_1}_{K'}$ and defines a noncut, which is impossible.

Now, suppose β is a noncut near e over $d^{i_2}_K$. Let $e = f(b')$. We may assume that β was the most recent element added to d^{i_1}, since if we prove the claim for that i_1, the claim shows it for all other i_1. Thus, if the index of β in d^{i_1} is γ, then $d^{i_1}_{<\gamma} = d_{J(\beta)}$. We then have $f^{-1}(\beta)$, which is a noncut over d_K near b', and thus must be $J(b')$-maximal. If $\beta \in d$, then (I3) prevents this from happening. Thus, $\beta \notin d$, and $\beta \in B$.

Suppose that, for some S', $K \sqsubset S' \sqsubset J(\beta)$, β is a noncut over $d_{S'}$. Then, by our construction, for some initial segment S, with $K \sqsubset S \sqsubset J(\beta)$, β is a noncut above 0 over d_S. But then β is necessarily a noncut above 0 over d_K, and, since b' is a cut over d_K, this is impossible.

Since $d_{J(\beta)} \frown \{\beta\}$ is decreasing, and β is a cut over d_K, necessarily $d_{J(\beta)}$ realizes only cuts over d_K – else, let j be the first element such that $\mathrm{tp}(d_j/d_{<j})$ is a noncut. Then d_j is necessarily strictly $<j$-maximal over $d_{>j}\beta$, but β is a cut over $d_{<j}$, contradiction.

We have $f^{-1}(\beta)$, a noncut near b' over d_K. Since $d_{J(\beta)}$ realizes only cuts over d_K, $f^{-1}(\beta)$ is a noncut near b' over $d_{J(\beta)}$ – note that $\mathrm{dcl}(d_{J(\beta)})$ contains no realizations of $\mathrm{tp}(b'/d_K)$, by (I3). But $b' = g(\beta)$, for some $d_{J(\beta)}$-definable g. Thus, $f^{-1}(\beta) - g(\beta)$ is a noncut over $d_{J(\beta)}$, while β is a cut over $d_{J(\beta)}$, contradiction. ∎

Claim 5.3.28. *Let i_1, i_2 be stages, $i_2 > i_1$. Let K be an initial segment of J^{i_2}, and let $K' = K \cap J^{i_1}$. If $\beta_1, \beta_2 \in d^{i_1}, d^{i_2}$, with $\beta_1, \beta_2 \notin d^{i_2}_K$, then $\beta_1 \prec_{d^{i_2}_K} \beta_2 \iff \beta_1 \prec_{d^{i_1}_{K'}} \beta_2$.*

Proof. Suppose the claim fails. WLOG, we may assume that β_1 "decreased" in the orderings, so either $\beta_1 \prec_{d_K^{i_2}} \beta_2$ but $\beta_1 \succsim_{d_{K'}^{i_1}} \beta_2$, or $\beta_1 \sim_{d_K^{i_2}} \beta_2$, but $\beta_1 \succ_{d_{K'}^{i_1}} \beta_2$. In either case, we can find a $d_K^{i_2}$-definable element, e, such that β_1 is a noncut near e, and $e \notin \mathrm{dcl}(d_{K'}^{i_1})$. But this contradicts Claim 5.3.27. $\square$

Claim 5.3.29. *d^i is decreasing, for every i.*

Proof. We go by induction. If d^i is not decreasing for i a successor stage, choose $K \sqsubset J^i$ witnessing the failure. Then, for any J' extending K, there is $j \in J'$, $k > j$, with $d_j^i \prec_K d_k^i$. Let b' be the element that was newly inserted at stage i.

If $d_k^i = b'$, then $d_{<k}^i = d_{<k}$, and, by Claim 5.3.23, $d_{\leq k}^i$ is decreasing, which is a contradiction. If $d_j^i = b'$, then $K \not=< j$, since d_j^i is strictly K-maximal. So then we can just take J' to not include j. Finally, we consider the case where $d_j^i, d_k^i \in d^{i-1}$. But this is impossible, by Claim 5.3.28.

If d^i is not decreasing at a limit stage, then choose $K \sqsubset J^i$ witnessing the failure. Then, for any J' extending K, there is $j \in J'$, $k > j$, with $d_j^i \prec_K d_k^i$. Let i_0 be the first stage at which d_j^i and d_k^i were both in d^{i_0}. Note that i_0 is a successor stage. But then, by Claim 5.3.28, d^{i_0} is not decreasing, contradicting induction. $\square$

Claim 5.3.30. *d^i satisfies* (I2), *for every i.*

Proof. Since, by construction, every element has (I2) at the stage it was first inserted, Claim 5.3.27 is enough. $\square$

Now, let $d' = \bigcup d^i$. By Claims 5.3.29 and 5.3.30, d' is decreasing and has (I2). We need only show

Claim 5.3.31. *d' satisfies* (I3).

Proof. Suppose not. Let K be an initial segment of d', with $a \in \mathrm{dcl}(d'_K) \setminus d'_K$, and a strictly J-maximal, with $a \notin \mathrm{dcl}(d'_J)$, for some $J \sqsubset K$. But note that we have only added elements of B to d to obtain d'. Thus, $a \in \mathrm{dcl}(dB) \setminus \mathrm{dcl}(d)$, and hence $a \in \mathrm{dcl}(B) = B$. Thus, since $a \notin d'_K$, there must be $b' \in B \cap d'_J$ – else a, or another element of B that was a cut over d_J, would have been inserted at some stage. Choose b' to be the first such inserted element. Let b' be strictly J'-maximal, for some $J' \sqsubset J$. Now, consider b' and a in our original sequence, d. Let $S = J \cap I^\alpha$. We show that a is strictly S-maximal, and S is the shortest such initial segment.

Since $\mathrm{tp}(a/d'_J)$ is a cut and $\mathrm{dcl}(d_S b') \supseteq d'_j$, we know $\mathrm{tp}(a/d_S b')$ is a cut. Since $\mathrm{tp}(b'/d_S)$ is a cut, this implies $\mathrm{tp}(a/d_S)$ is a cut. Suppose some element of $d_{>S}$ is a cut over d_S. By Claim 5.3.27, it would then be a cut over d_J, contradicting a being strictly J-maximal. Thus, a is strictly S-maximal.

But now, by Claim 5.3.22, it is not possible for a to be strictly $\prec_{d_S b'}$-maximal over $d_{>S}$, contradiction. $\square$

Then, setting $c^{\alpha+1} = d'$, we are done with this stage.

Limits

We have shown that, at each successor stage, c^α continues to satisfy the three induction conditions. We must now show that the induction conditions are satisfied at limit stages. First, some useful results.

Claim 5.3.32. *Let $\beta \in c^{i_1}$, for some stage i_1. Let $i_2 > i_1$. For $K \sqsubset J^{i_2}$, let $K' = K \cap I^{i_1}$. Then β is a noncut near $e \in \mathrm{dcl}(c^{i_2}_K)$ over $c^{i_2}_K$ iff β is a noncut near e over $c^{i_1}_{K'}$ (in particular, $e \in \mathrm{dcl}(c^{i_1}_{K'})$). Moreover, the reverse direction is true for any $\beta \in N$, not just $\beta \in c^{i_1}$.*

Proof. First, the forward direction. For a contradiction we may assume that $c^{i_2}_K$ realizes e, a noncut near β (over $c^{i_1}_{K'}$) that $c^{i_1}_{K'}$ does not. WLOG, we may assume that i_2 is the first stage at which e is $c^{i_2}_K$-definable, and then assume that i_1 is the previous stage, so $i_2 = i_1 + 1$. Let b be the element that was added to c^{i_1} to obtain c^{i_2}. Then b is strictly $J(b)$-maximal, for some $J(b) \sqsubset K'$. Then $e = f(b)$, where f is $c^{i_1}_K$-definable. But then $f^{-1}(\beta)$ must be a noncut near b over $c^{i_1}_K$, and hence over $c^{i_1}_{J(b)}$, and is therefore strictly $J(b)$-maximal. By (I3) for stage i_1, this is impossible.

Now the reverse direction – it proceeds exactly as in Claim 5.3.27. We do it for arbitrary $\beta \in N$. Suppose β is a noncut near e over $c^{i_1}_{K'}$. If β is not a noncut near e over $c^{i_2}_K$, then there is some $\mathrm{dcl}(c^{i_2}_K)$-definable e' between e and β. WLOG, we may assume that i_2 is the first stage at which such an e' is $c^{i_2}_K$-definable, and then assume that i_1 is the previous stage, so $i_2 = i_1 + 1$. Let b be the element that was added to c^{i_1} to obtain c^{i_2}. We may assume that $e' = f(b)$, where f is a $c^{i_1}_{K'}$-definable function. We know e' and β must have the same (noncut) type over $c^{i_1}_{K'}$, since $e' \notin \mathrm{dcl}(c^{i_1}_{K'})$. But b realizes a cut over $c^{i_1}_{K'}$ (by Claims 5.3.18 and 5.3.20), and defines a noncut, which is impossible. ◻

Claim 5.3.33. *Let $i_1 < i_2 < \lambda$, and $\beta_1, \beta_2 \in c^{i_1}, c^{i_2}$. Let K be an initial segment of I^{i_2}, with $K' = K \cap I^{i_1}$. Then $\beta_1 \prec_{c^{i_1}_{K'}} \beta_2$ iff $\beta_1 \prec_{c^{i_2}_K} \beta_2$.*

Proof. As in Claim 5.3.28, for a contradiction we may assume that $c^{i_2}_K$ realizes e, a noncut near β_1 (over $c^{i_1}_{K'}$) that $c^{i_1}_{K'}$ does not. But that is impossible by Claim 5.3.32. ◻

Now we are ready to show that the induction conditions apply to c^λ, λ a limit. Let $d = c^\lambda$.

Claim 5.3.34. *d is decreasing.*

Proof. Assume not, so choose $K \sqsubset I^\lambda$ witnessing the failure. Then, for any J' extending K, there is $j \in J'$, $k > j$, with $d_j \prec_K d_k$. Let i be the first stage at which d_j and d_k were both in d^i. Note that i_0 is a successor stage. But then, by Claim 5.3.33, d^i is not decreasing, contradicting induction. ◻

Claim 5.3.35. *d satisfies* (I2).

Proof. Since, by construction, every element has (I2) at the stage it was first inserted, Claim 5.3.32 is enough. ◻

Claim 5.3.36. *d satisfies* (I3).

Proof. Suppose not. Let K be an initial segment of d, with $a \in \mathrm{dcl}(d_K) \setminus d_K$, and a strictly J-maximal, with $a \notin \mathrm{dcl}(d_J)$, for some $J \sqsubset K$. Let i be the first stage at which a is $c^i_{I^i \cap K}$-definable, and let $K' = K \cap I^i$. If a were strictly $J' = J \cap I^i$-maximal (over c^i), then we would have a contradiction, since c^i satisfies (I3).

$\mathrm{tp}(d_j/d_J)$ is a noncut, for $j > J$, $j \in I^\lambda$. Thus, $\mathrm{tp}(d_j/d_{J'})$ is a noncut, for $j > J'$, $j \in I^i$. But by the reverse direction of Claim 5.3.32, with the "moreover" clause, $\mathrm{tp}(a/d_{J'})$ is a cut, and hence a is strictly J'-maximal, contradiction. □

Definable Closure

At some point, the above construction halts. This must be because $N \setminus \mathrm{dcl}(c^\alpha)$ is empty, for some α. Let $c' = c^\alpha$. Note that c' satisfies the three induction conditions. Now we show that we can insert every element of $\mathrm{dcl}(c')$ into c' while preserving the induction conditions. We start with $c'^0 = c'$, and transfinitely insert elements, preserving the decreasing condition and (I3). First, we show that $c'^{\alpha+1}$ satisfies these two conditions. Let $d = c'^\alpha$. Let $b \in \mathrm{dcl}(d)$.

Claim 5.3.37. *$b \in \mathrm{dcl}(d_{J(b)})$.*

Proof. True by (I3) for d. □

Let $d' = c'^{\alpha+1}$.

Claim 5.3.38. *d' is decreasing.*

Proof. Clearly, any counterexample would have to come for an initial segment K such that d'_K contained b. But no new elements are definable with b, since $b \in \mathrm{dcl}(d_{J(b)})$, so we are done. □

Claim 5.3.39. *d' satisfies* (I3)*.*

Proof. Clear – $\mathrm{dcl}(d') \setminus d' \subset \mathrm{dcl}(d) \setminus d$, so an element of $\mathrm{dcl}(d) \setminus d$ would have to become strictly J-maximal (or not d_J-algebraic) for some J when it was not before, which is impossible. □

For limit stages, similar (but easier) arguments as for the previous limit stage arguments work. So we are done. □

Chapter 6

Extending Continuous Functions to Closed Sets

6.1 Exploring the Question

Given a bounded definable function and a definable set on which the function is continuous, we may ask what happens when we try to extend the function continuously to the closure of the set.

Remark 6.1.1. For any o-minimal structure, M, and any 1-type, p, if f is a bounded M-definable function, there is a definable set $C \in p$, such that f is continuous on C, and f can be extended continuously to the closure of C, $\overline{C}$.

Proof. By cell decomposition, we can partition M into a finite number of intervals and points, on each of which f is continuous. One of these intervals or points must be in p. If a point is in p, then that point is an acceptable C. If p is the noncut near $\pm\infty$, then the rightmost (leftmost) interval in the partition is contained in p, so we may assume that p is not near $\pm\infty$. Thus, we may assume that there is an interval $(a_1, a_2) \in p$ such that f is continuous on (a_1, a_2), with $a_1, a_2 \in M$. Since f is continuous and bounded in an interval above a_1, $\lim_{x \to a_1^+} f(x)$ exists, as does $\lim_{x \to a_2^-} f(x)$, by [vdD98], Chapter 3, Corollary 1. Then let $\overline{f}$ be defined by

$$\overline{f}(x) = \begin{cases} f(x) & x \in (a_1, a_2) \\ \lim_{x \to a_1^+} f(x) & x = a_1 \\ \lim_{x \to a_2^-} f(x) & x = a_2 \end{cases} .$$

$\overline{f}$ is continuous, since continuity at $x \in (a_1, a_2)$ is given by continuity of f on (a_1, a_2), and continuity at a_i, $i = 1, 2$, is given by the construction of $\overline{f}$ at those points. $\square$

The analogue of Remark 6.1.1 for higher dimensions is false in general.

Example 6.1.2. Let $M = (\mathbb{R}, +, \cdot, <, 0, 1)$, the reals as an ordered field. Let

$$f(x, y) = \begin{cases} \frac{y}{x} & y \leq x \\ 1 & O.W. \end{cases} ,$$

with domain $\{\langle x, y\rangle \mid x, y > 0\}$. Then f is continuous and bounded on its domain, but it cannot be continuously extended to the origin, which is in the closure of its domain, since on the line $L_m = \{\langle x, y\rangle y = mx\}$, f has a constant value of m, and the closure of each L_m includes the origin.

However, while we may not be able to extend a function to the closure of its entire domain, we can extend the function to the closure, provided we restrict the domain appropriately.

Example 6.1.3. Let M and f be as above. If we restrict the domain of f to $\{\langle x, y\rangle \mid y < x^2\}$, then f can be extended continuously to the closure of its domain.

Lemma 6.1.4. *Let M expand a real closed field, and let f be an M-definable function, continuous and bounded on its domain, C. Then, for any $d \in \bar{C}$, there is some $C' \subseteq C$ such that $d \in \overline{C'}$ and f extends continuously to $\overline{C'}$.*

Proof. This is an easy corollary of the main result that we prove later. □

However, we can ask a more general question. Instead of asking only for a point to be in the closure of the domain, we can ask about membership of a type in the domain. Here, we fail in a more subtle way.

Example 6.1.5. Let $M = (\mathbb{R}, +, \cdot, <)$, the reals as an ordered field. Let $p(x_1, x_2)$ be the type generated by the formulas $x_1 > 0$, $x_1 < 1/n$, for $n = 1, 2, \ldots$, $0 < x_2 < x_1$, $x_2 < ax_1$, and $ax_1^q < x_2$, for $a \in \mathbb{R}_+$, $q \in \mathbb{Q}_{>1}$.

Claim 6.1.6. *p is consistent.*

Proof. Let Γ be any finite subset of the above formulas. Let

$$\begin{aligned} m_1 &= \frac{\min\{1/n \mid x_1 < 1/n \in \Gamma\}}{2} \\ a &= \frac{\min\{r \mid x_2 < rx_1 \in \Gamma\}}{2} \\ m_2 &= \frac{\min\left\{x \mid rx_1^q < x_2 \in \Gamma \Rightarrow \frac{a}{r} = x^{q-1}\right\}}{2}. \end{aligned}$$

Note that m_2 is well-defined, since $a, r, q-1 > 0$. Then, if $b = \min(m_1, m_2)$, (b, ab) satisfies Γ, since for each formula $x_1 < 1/n$, $b \leq m_1$ guarantees that it is satisfied, for each formula $x_2 < rx_1$, our choice of a guarantees that it is satisfied, and for each formula $rx_1^q < x_2$, $b \leq m_2$ guarantees that $b^{q-1} < a/r$, so $rb^q < ab$. □

Claim 6.1.7. *p determines a complete 1-type on the first coordinate, x_1.*

Proof. Since the sequence $\{1/n \mid n \in \mathbb{N}\}$ is cofinal in $\mathbb{R}$ towards 0, we know that $x_1 < r$, for every $r \in \mathbb{R}$, $r > 0$, as well, we know that $x_1 > 0$, and thus $x_1 > r$, for every $r \in \mathbb{R}$, $r \leq 0$. Since order type determines type, we then have that the type of x_1 is completely determined. □

Claim 6.1.8. *If $g(x)$ is any M-definable function, with $\lim_{x\to 0^+} g(x) = 0$, then, for some $s \in \mathbb{R}$, and some $q \in \mathbb{Q}$,*

$$\lim_{x\to 0^+} \frac{g(x)}{sx^q} = 1.$$

Proof. By quantifier elimination for M (Theorem 1.3.6), we know that g is given by a quantifier-free formula $\psi(x, y)$, such that, for any x, $\psi(x, y)$ holds iff $y = g(x)$. We may assume that, for x sufficiently close to 0, ψ is a conjunction of atomic formulas – by [Hod93], section 2.3, ψ can be put in disjunctive normal form, and a particular element of the disjunction must hold in a neighborhood above 0 by o-minimality. Let $\theta_1, \ldots, \theta_k$ be the atomic formulas that are equalities in ψ, with θ_i given by $t_i^1 = t_i^2$, where t_i^1, t_i^2 are terms in variables x, y. Then note that the formula $\sum_{i=1}^k (t_i^1 - t_i^2)^2 = 0$ is satisfied if and only if $\bigwedge_{i=1}^k \theta_i$ is, so we may assume that there is exactly one equality in ψ. Cross-multiplying, and dividing through by common factors of x and y, we may assume that the equality consists of a polynomial in x and y, with at least one term being solely a power of x, and one term being solely a power of y. Rearrange so that the lowest power of y is alone on the left-hand side of the equation. Then we can write the equality as

$$y^m = \sum_{i=1}^n c_i x^{k_i} y^{l_i},$$

with $c_i \in \mathbb{R}$, $c_i \neq 0$, $k_i, l_i \in \mathbb{Z}_{\geq 0}$, with $(k_i, l_i) \neq (k_j, l_j)$ for $i \neq j$, and if $k_i = 0$, then $l_i > m$. Define the functions

$$h_i(x) = x^{k_i} g(x)^{l_i}, \qquad i = 1, \ldots, m.$$

In a neighborhood above 0, there must be a between 1 and m such that $h_a(x) > h_i(x)$, $i = 1, \ldots, a-1, a+1, \ldots, m$, by o-minimality – if there were equality for h_i and h_j, then, since $(k_i, l_i) \neq (k_j, l_j)$, we would obtain a direct representation of g as a fractional power of x. Let

$$b_i = \lim_{x\to 0^+} \frac{h_i(x)}{h_a x}, \qquad i = 1, \ldots, a-1, a+1, \ldots, m.$$

First, suppose that some $b_i > 0$. Then we have

$$1 \geq \lim_{x\to 0^+} \frac{x^{k_i} g(x)^{l_i}}{x^{k_a} g(x)^{l_a}} = \lim_{x\to 0^+} \left(\frac{g(x)}{x^{\frac{k_a - k_i}{l_i - l_a}}} \right)^{l_i - l_a} > 0.$$

Note that $l_i - l_a \neq 0$, since if it were, we would have $\lim_{x\to 0^+} 1 \leq x^{k_a - k_i} > 0$, and $k_a \neq k_i$, which is impossible.

If $l_i - l_a > 0$, then

$$1 \geq \lim_{x\to 0^+} \frac{g(x)}{x^{\frac{k_a - k_i}{l_i - l_a}}} > 0,$$

since $t^{l_i - l_a}$ is continuous at and above 0. If $l_i - l_a < 0$, then since the expression is bounded away from 0, there must be some $N \in \mathbb{N}$ such that

$$1 \leq \lim_{x\to 0^+} \frac{g(x)}{x^{\frac{k_a - k_i}{l_i - l_a}}} < N.$$

In either case, we may find $c \in \mathbb{R}$, such that

$$\lim_{x\to 0^+} \frac{g(x)}{cx^{\frac{k_a-k_i}{l_i-l_a}}} = 1,$$

proving the claim.

Thus, we may assume that $b_i = 0$, for all i. Restrict to an interval above 0 such that

$$\frac{h_i(x)}{h_a x} < \frac{c_a}{2n|c_i|},$$

for all i. Then we know

$$\sum_{\substack{i=1\\ i\neq a}}^{n} c_i h_i(x) \leq \sum_{\substack{i=1\\ i\neq a}}^{n} |c_i| h_i(x) < \sum_{\substack{i=1\\ i\neq a}}^{n} |c_i| \frac{c_a h_a(x)}{2n|c_i|} < \frac{c_a}{2} h_a(x),$$

and also

$$\sum_{\substack{i=1\\ i\neq a}}^{n} c_i h_i(x) \geq \sum_{\substack{i=1\\ i\neq a}}^{n} -|c_i| h_i(x) \geq \sum_{\substack{i=1\\ i\neq a}}^{n} -|c_i| \frac{c_a h_a(x)}{2n|c_i|} > -\frac{c_a}{2} h_a(x).$$

Thus,

$$\begin{aligned}
-\frac{c_a}{2} h_a(x) &< \sum_{\substack{i=1\\ i\neq a}}^{n} c_i h_i(x) < \frac{c_a}{2} h_a(x) \\
c_a h_a(x) - \frac{c_a}{2} h_a(x) &< \sum_{i=1}^{n} c_i h_i(x) < c_a h_a(x) + \frac{c_a}{2} h_a(x) \\
\frac{c_a}{2} h_a(x) &< \sum_{i=1}^{n} c_i h_i(x) < \frac{3c_a}{2} h_a(x) \\
\frac{c_a}{2} h_a(x) &< \sum_{i=1}^{n} c_i x^{k_i} g(x)^{l_i} < \frac{3c_a}{2} h_a(x) \\
\frac{c_a}{2} h_a(x) &< g(x)^m < \frac{3c_a}{2} h_a(x) \\
\frac{c_a}{2} x^{k_a} g(x)^{l_a} &< g(x)^m < \frac{3c_a}{2} x^{k_a} g(x)^{l_a} \\
\frac{c_a}{2} x^{k_a} &< g(x)^{m-l_a} < \frac{3c_a}{2} x^{k_a} \\
\frac{c_a}{2} &< \frac{g(x)^{m-l_a}}{x^{k_a}} < \frac{3c_a}{2}.
\end{aligned}$$

Note that we have $l_a \neq m$, since otherwise we would have $1 < \frac{3c_a}{2} x^{k_a}$ for x in a neighborhood of 0, which cannot be true unless $k_a = 0$, in which case we have $l_i > m$. Then

$$\begin{aligned}
\frac{c_a}{2} &< \frac{g(x)}{x^{k_a/(m-l_a)}} < \frac{3c_a}{2} \\
\frac{c_a}{2} &< \lim_{x\to 0^+} \frac{g(x)}{x^{k_a/(m-l_a)}} < \frac{3c_a}{2}.
\end{aligned}$$

By o-minimality, this limit exists, and by the inequalities, is not 0 or $\pm\infty$. Denote it by d. Then

$$\lim_{x\to 0^+} \frac{g(x)}{dx^{k_a/(m-l_a)}} = 1,$$

proving the claim. ☐

Let C be any cell with $C \in p$. We may assume that C is of the form

$$C = \{\langle x, y\rangle \mid 0 < x < c \wedge f(x) < y < g(x)\},$$

for some $c \in \mathbb{R}$ and some $\mathbb{R}$-definable f, g.

Claim 6.1.9. *There exist $d_1, d_2 \in \mathbb{R}, q_1, q_2 \in \mathbb{Q}_+$, with $q_1 > 1 \geq q_2$, such that*

$$\lim_{x\to 0^+} \frac{f(x)}{d_1 x^{q_1}} = 1$$
$$\lim_{x\to 0^+} \frac{g(x)}{d_2 x^{q_2}} = 1.$$

Proof. By Claim 6.1.8, the existence of some d_1, d_2 and q_1, q_2 is guaranteed. It remains to show that $q_1 > 1 \geq q_2$. First, for a contradiction, assume $q_1 \leq 1$. Note that we can certainly assume that $f(x) \geq 0$. Then

$$1 = \lim_{x\to 0^+} \frac{f(x)}{d_1 x^{q_1}} \leq \lim_{x\to 0^+} \frac{f(x)}{d_1 x},$$

so for sufficiently small x, $d_1 x \leq f(x)$, but since then $p \models y < d_1 x \leq f(x)$, it is impossible that $C \in p$. Thus, $q_1 > 1$.

Now we show that $q_2 \leq 1$. Suppose $q_2 > 1$. Then we have

$$1 = \lim_{x\to 0^+} \frac{g(x)}{d_2 x^{q_2}}.$$

We know that $g(x) > 0$ for $x > 0$, so $d_2 \neq 0$. Thus, for sufficiently small x,

$$2d_2 x^{q_2} > g(x).$$

But since then $p \models y > 2d_2 x^{q_2} > g(x)$, it is impossible that $C \in p$. Thus, $q_2 \leq 1$. ☐

Claim 6.1.10. *If C is any set with $C \in p$, with the function on the first quadrant*

$$F(x,y) = \begin{cases} \frac{y}{x} & y < x \\ 1 & \text{Otherwise} \end{cases},$$

continuous on C, then F does not extend continuously to $\overline{C}$.

Proof. Without loss of generality, we may assume that C is a cell, with boundaries in the y-coordinate given by $\mathbb{R}$-definable f, g. We may assume that $g(x) \leq x$, so $g(0) = 0$, and that $f(x) \geq 0$, so $f(0) = 0$. Then by Claim 6.1.9, we know that there exist $d_1, d_2 \in \mathbb{R}$, $q_1 > 1 \geq q_2 \in \mathbb{Q}$ such that

$$\lim_{x \to 0^+} \frac{f(x)}{d_1 x^{q_1}} = 1$$
$$\lim_{x \to 0^+} \frac{g(x)}{d_2 x^{q_2}} = 1.$$

Thus, we can restrict to an interval above 0 such that

$$\frac{d_1}{2} x^{q_1} \leq f(x) \leq 2 d_1 x^{q_1}$$
$$\frac{d_2}{2} x^{q_2} \leq g(x) \leq 2 d_2 x^{q_2}.$$

For $x > 0$, it is easy to see that F must extend to $\langle x, f(x) \rangle$ and $\langle x, g(x) \rangle$ in the natural way, as $F(x, f(x)) = \lim_{y \to f(x)^+} F(x, y)$, and similarly with $\langle x, g(x) \rangle$, since the value of F along any curve approaching $\langle x, f(x) \rangle$ must have the same limit, by [vdD98], Chapter 6, 4.2. Then for x in this interval above 0,

$$\frac{d_1}{2} x^{q_1 - 1} \leq F(x, f(x)) \leq 2 d_1 x^{q_1 - 1}$$
$$\min\left(1, \frac{d_2}{2} x^{q_2 - 1}\right) \leq F(x, g(x)) \leq \min\left(2 d_2 x^{q_2 - 1}\right).$$

But

$$\lim_{x \to 0^+} 2 d_1 x^{q_1 - 1} = 0 = \lim_{x \to 0^+} \frac{d_1}{2} x^{q_1 - 1}$$
$$\min\left(2 d_2 x^{q_2 - 1}\right) \geq \min\left(1, \frac{d_2}{2} x^{q_2 - 1}\right) > 0,$$

so

$$\lim_{x \to 0^+} F(x, f(x)) = 0 < \lim_{x \to 0^+} F(x, g(x)).$$

Since both the curves $\langle x, f(x) \rangle$ and $\langle x, g(x) \rangle$ are in $\overline{C}$, and they both have the origin as their limit point, we have, again by [vdD98], Chapter 6, 4.2, that F can extend continuously to the origin only if F has the same limit along the two curves. But it does not, and since $\overline{C}$ includes the origin, F cannot be continuously extended to $\overline{C}$. $\square$

We may ask, then, for necessary and sufficient conditions on p, an n-type in an o-minimal structure, we give necessary and sufficient conditions on p so that, for any F, a bounded definable function, there exists a definable set, C, such that: C contains any realization of p; F is continuous on C; and F can be continuously extended to C's closure.

6.2 Good Bounds and i-Closures

We will be helped in answering our question by some technical results and lemmas concerning the closures of sets. In this section, we assume that all noncuts are interdefinable over the empty set.

Condition 6.2.1. We will be working under the following assumptions for the rest of this section. Let p be a decreasing independent n-type over a set A, c a realization of p, i an index in p's coordinates, and $k = N(i) > 0$ ($N(i)$ is from Definition 5.1.21). As well, we assume that $\operatorname{tp}(c_j/c_{<k}A)$ is a noncut near $\beta_j(c_{<k}) \neq \pm\infty$ for $j \geq k$. Denote $\beta = \langle \beta_k, \ldots, \beta_n \rangle$.

Note that if Condition 6.2.1 is true for some c, it is true for any $c_0 \models p$, and thus can also be thought of as a condition just on p, A, i, and k. As well, note that, for any p a decreasing type over A and $c \models p$ with $k = N(i)$ for some coordinate i, $\operatorname{tp}(c_j/c_{<k}A)$ is a noncut by Corollary 5.1.22.

Lemma 6.2.2. *If p, A, c, i, and k satisfy Condition 6.2.1, then there is an A-definable set, C^0, containing c such that, for every $a \in \pi_{\leq k-1}(C^0)$, $\overline{C^0}$ contains a unique point, d, with $d_{\leq k} = \langle a, \beta_k(a) \rangle$. Moreover, for each a (and in particular for $c_{<k}$), this point is independent of choice of C^0 – in fact, it is $\langle a, \beta(a) \rangle$.*

Proof. We assume that, for $j \geq k$, c_j is a noncut above $\beta_j(c_{<k})$ – the proof is not affected by this.

By Lemma 2.2.10, for each $j > k$ there is some A-definable k-ary function, h_j, such that

$$\begin{gathered} c_j < h_j(c_{\leq k}), \text{ and} \\ \lim_{x \to \beta_k(c_{<k})} h_j(c_{<k}, x) = \beta_j(c_{<k}). \end{gathered} \tag{6.1}$$

Let C be an A-definable set containing c such that β is continuous on C, $h_j > \beta_j(c_{<k})$ for $j \geq k$ (possible since $h_j(c_{\leq k}) > \beta_j(c_{<k})$), and (6.1) holds on all of C (possible since it holds for c – note that the limit statement is first-order). Let

$$B = \{x \mid \forall j \geq k(x_j > \beta_j(x_{<k})) \wedge \forall j > k(x_j < h_j(x_{\leq k}))\}.$$

Let $C' = C \cap B$. Note that, since $c \in C$, $c \in B$, we know C' is non-empty. Now, by Lemma 1.3.14, we can decompose C' into definable sets, on each of which, for any $a \in \pi_{\leq k-1}(C')$, $\overline{C'_a} = \overline{C'}_a$ – the closure of a fiber is the fiber of the closure. Let C^0 be the set containing that fiber.

Let $a \in \pi_{\leq k-1}(C^0)$. Let $D = \{a\} \times C^0_a$. Let $d \in \overline{C^0}$, with $d_{\leq k} = \langle a, \beta_k(a) \rangle$. Note that this implies $d \in \overline{D}$. We want to show that $d = \langle a, \beta(a) \rangle$. Let $\gamma(t)$ be an Aa-definable curve in D, with $\gamma(0) = d$, and $\operatorname{dom}(\gamma) = (0, s)$, for some positive s. Then, for $j > k$,

$$d_j \geq \lim_{t \to 0^+} \beta_j(\gamma(t)_{<k}) = \beta_j(a).$$

Similarly,

$$d_j \leq \lim_{t \to 0^+} h_j(\gamma(t)_{\leq k}) = \lim_{y \to \beta_j(a)^+} h_j(a, y) = \beta_j(a).$$

Thus, $d = \langle a, \beta(a) \rangle$. ∎

Definition 6.2.3. Assume Condition 6.2.1 holds. Then, for any tuple a with length at least $k-1$, such that $a_{<k} \in \pi_{\leq k-1}(C^0)$, let

$$\mathrm{icl}_p(i,a) = \langle a_{<k}, \beta(a_{<k})\rangle.$$

When p is clear from context, we may omit it, writing simply $\mathrm{icl}(i,a)$, and also referring to this as the *i-closure of a*.

Note that $\mathrm{icl}(i,a)$ is an $Aa_{<k}$-definable point.

Lemma 6.2.4. *Assume Condition 6.2.1 holds. If* $\mathrm{tp}(c_i/Ac_{<i})$ *is a cut, then* $\mathrm{icl}(i,x) = \mathrm{icl}(i-1,x)$.

Proof. If $\mathrm{tp}(c_i/Ac_{i-1})$ is a cut, then, by definition, $N(i) \neq i$, so $N(i) \leq i-1$. Since now the conditions on $N(i)$ and $N(i-1)$ are identical, $N(i) = N(i-1)$, and so

$$\mathrm{icl}(i,x) = \langle x_{<N(i)}, \beta_{N(i)}(x_{<N(i)}), \ldots, \beta_n(x_{<N(i)})\rangle = \mathrm{icl}(i-1,x).$$

□

Definition 6.2.5. Assume Condition 6.2.1 holds. Let f be an i-ary A-definable function such that, for some A-definable C with $c \in C$, f is continuous (as a function of the first i coordinates), non-negative, and f extends to $\overline{C}$ such that $f(\mathrm{icl}(i,x)) = 0$, for all $x \in C$. Then we call f a *good bound at i*.

Note that the set of good bounds at i (for a given p) forms a vector space over A. As well, note that, if $c_{N(i)}$ is a noncut near $\beta_{N(i)}(c_{<N(i)}) \neq \pm\infty$, then

$$m_i(x_{\leq i}) = |x_{N(i)} - \beta_{N(i)}(x_{<N(i)})|$$

is a good bound at i.

Lemma 6.2.6. *Assume Condition 6.2.1 holds. If f is a good bound at i then there exists f' such that $f' \geq f$ on some definable set containing c, and f' is a good bound at $i-1$.*

Proof. Note that there must be a noncut at or before the $i-1$ coordinate in p, otherwise the "good bound" condition is vacuous.

We first consider the case where $\mathrm{tp}(c_i/c_{<i}A)$ is a noncut. By the definition of a good bound, there is some A-definable C such that f is continuous and non-negative on C, and f extends to $\overline{C}$ such that $f(\mathrm{icl}(i,x)) = 0$ for all $x \in C$. Since c_{i-1} is a noncut near some $Ac_{<N(i-1)}$-definable element, assume c_{i-1} is a noncut above (WLOG) $\beta_{i-1}(c_{<N(i-1)})$, where β_{i-1} is A-definable (note that β_{i-1} is not part of the original sequence of functions, β). We may restrict C so that $\mathrm{icl}(i-1,x) \notin C$, since we can take C to have lower boundary at least $\beta_{i-1}(x_{<N(i-1)})$ at the $i-1$st coordinate. Note that, on C, $\mathrm{icl}(i-1,x) \neq \mathrm{icl}(i,x)$, since $\mathrm{icl}(i-1,x)_{i-1} = \beta_{i-1}(x_{<N(i-1)}) < x_{i-1} = \mathrm{icl}(i,x)_{i-1}$. Since f is a good bound at i, we know that $f(\mathrm{icl}(i,x)) = 0$ for $x \in C$, and therefore $f(\mathrm{icl}(i,x)) < m_{i-1}(\mathrm{icl}(i,x))$. Assume WLOG that c_i is a noncut above some $\alpha(c_{<i})$, for some A-definable α. Thus, for each x, there is some $h(x_{<i})$ such that, if $x_i \in (\alpha(x_{<i}), h(x_{<i}))$, $f(x) < m_{i-1}(x)$. Restrict C to have upper boundary at most h on the ith coordinate. Then, on our new C, $m_{i-1} > f$, and m_{i-1} is a good bound at $i-1$.

Now consider the case where $\mathrm{tp}(c_i/c_{<i}A)$ is a cut. There is a closed $Ac_{<i}$-definable interval, $J(c_{<i})$, about c_i on which $f(c_{<i}, -)$ is continuous. Thus, for $x_{<i}$ in some A-definable set containing c, say C'', there is $J(x_{<i})$, a closed $Ax_{<i}$-definable interval such that $f(x_{<i}, -)$ is continuous. Let $C' = \{x \in C'' \mid x_i \in J(x_{<i})\}$, an A-definable set.

We can then let $f'(x_{<i}) = \sup\{f(x_{\leq i}) \mid x_i \in J(x_{<i})\}$. Clearly, $f(x_{\leq i}) < f'(x_i)$. We must also show that f' is a good bound at $i-1$ – that is, f' extends to $\mathrm{icl}(i-1, x)$ by 0, for $x \in C'$. Since f extends to $\mathrm{icl}(i, x) = \mathrm{icl}(i-1, x)$ by 0, we know that, for any definable curve in C with limit point $\mathrm{icl}(i, x)$, $f(x_{\leq i})$ goes to 0 on the curve. This implies that $\sup\{f(x_{\leq i}) \mid x_i \in J(x_{<i})\}$ also goes to 0 on the curve – suppose not. Then there is a curve, γ, and $\epsilon > 0$ such that, for each $t > 0$, there is an $x_i \in J(\gamma(t)_{<i})$ such that $f(\gamma(t)_{<i}, x_i) > \epsilon$. We can definably choose x_i as a function of t (and ϵ), thus yielding a new curve, γ', with $f(\gamma'(t))$ not going to 0, contradiction. □

6.3 Main Result

Theorem 6.3.1. *Let T be such that all noncuts are interdefinable over the empty set (e.g., T expands the theory of an ordered field). Let A be a set. Let p be a decreasing n-type over A. Let $c = \langle c_1, \ldots, c_n\rangle \models p$. The following statements are equivalent.*

1. *For every A-definable function, F, there is an A-definable set, C, such that $c \in C$, F is continuous on C, and F extends continuously to $\overline{C}$.*

2. *For $i = 1, \ldots, n$, $\mathrm{tp}(c_i/Ac_{<i})$ is algebraic, a noncut, a uniquely realizable cut, or an out of scale nonuniquely realizable cut on A.*

We first prove the "if" of Theorem 6.3.1.

Proposition 6.3.2. *Let T be such that all noncuts are interdefinable over the empty set, let A be a set, and let p be a decreasing n-type over A. Let $c = \langle c_1, \ldots, c_n\rangle$ realize p. Suppose that, for $i = 1, \ldots, n$, $\mathrm{tp}(c_i/c_{<i-1}A)$ is not in scale or near scale on A (i.e., $\mathrm{tp}(c_i/c_{<i-1}A)$ is algebraic, a noncut, a uniquely realizable cut, or an out of scale nonuniquely realizable cut on A). Then, for any F a bounded A-definable function on M^n, there is an A-definable set, C, such that F is continuous on C, F can be continuously extended to $\overline{C}$, and $c \in \overline{C}$.*

Proof. [1]

We will go by induction on n, although we will also have an additional "inner" induction. Note that, by absorbing A into our language, we may assume that $A = \emptyset$. Henceforth in the proof, "definable" means "$\emptyset$-definable," unless otherwise indicated. Let P be the prime model of T.

Regularizing noncuts

Let the coordinates at which $\mathrm{tp}(c_i/c_{<i})$ is a noncut be $i_1, \ldots, i_l$, and let $I = \{i_1, \ldots, i_l\}$. Let the function of $c_{<i}$ over which c_i is a noncut be α_i.

[1] The use of van den Dries' result on fiberwise-continuous functions is based on the proof in [Spe08].

Claim 6.3.3. *It is sufficient to prove Proposition 6.3.2 in the case where no c_i is a noncut near $\pm\infty$ over $\emptyset$.*

Proof. Let $I_\infty^+ = \{i \in I \mid \mathrm{tp}(c_i) = \infty^-\}$ and $I_\infty^- = \{i \in I \mid \mathrm{tp}(c_i) = -\infty^+\}$. Let $q^\pm$ be the map taking the noncut near $\pm\infty$ to the noncut above 0. Then we can consider the map $\xi(x_1, \ldots, x_n) = \langle y_1, \ldots, y_n\rangle$, with $y_i = q^+(x_i)$ if $i \in I_\infty^+$, $y_i = q^-(x_i)$ if $i \in I_\infty^-$, and x_i otherwise. Note that $\xi^{-1} = \xi$.

We assume that there is a definable set C' containing $\xi(c)$ such that $F \circ \xi$ is continuous on C' and extends continuously as $\overline{F \circ \xi}$ to $\overline{C'}$, and prove Proposition 6.3.2, giving the claim. Replace c by $\xi(c)$. Since every $|c_i|$ is bounded by some $r_i \in P$, we may assume C' is bounded, say $|x|_m < r$, for every $x \in C'$, with $|\cdot|_m$ the sup norm. Let $B = P \cap \mathrm{dom}(\xi)$. Let $C'' = \overline{C'} \cap B$. Note that C'' is closed in B. Let $C = \xi(C'')$. Since ξ is continuous on B and is also its own inverse on $\xi(B)$, it is a homeomorphism between B and $\xi(B)$. Note that C is closed in $\xi(B)$, since ξ is a homeomorphism. Moreover, C is closed in P^n, since $\xi(B) = P^n \setminus \{x \in P^n \mid \exists i \in I_\infty(x_i = 0)\}$, and since $x \in C''$ implies $|x_i| < r$, for every $i = 1, \ldots, n$, then $x \in C$ implies $|x_i| > 1/r$ for every $i = 1, \ldots, n$, so $\overline{C} \cap (P \setminus \xi(B)) = \emptyset$.

We claim that F has a continuous extension, $\overline{F}$, on C, defined by $\overline{F}(x) = (\overline{F \circ \xi} \circ \xi)(x)$. To prove $\overline{F}$ is continuous, take $D \subset P$ closed; we wish to show that $\overline{F}^{-1}(D) \cap C$ is closed. Since ξ is a homeomorphism, this is equivalent to asking if $\xi(\overline{F}^{-1}(D) \cap C)$ is closed. Note that $a \in C \iff \xi(a) \in C''$, so we may ask if $\xi(\overline{F}^{-1}(D)) \cap C''$ is closed. We have $(\xi \circ \overline{F}^{-1})(D) \cap C'' = \xi \circ (\overline{F \circ \xi} \circ \xi)^{-1}(D) \cap C'' = \xi \circ \xi \circ \overline{F \circ \xi}^{-1}(D) \cap C'' = \overline{F \circ \xi}^{-1}(D) \cap C''$, which is closed by continuity of $F \circ \xi$ on C''. Thus, F is continuous on C, proving Proposition 6.3.2. $\square$

Thus, we may assume that no α_i is $\pm\infty$.

By the same method, using the map $\xi'(x_1, \ldots, x_n) = \langle y_1, \ldots, y_n\rangle$, with $y_i = -x_i$ if $i \in I$ and c_i is a noncut below α_i, and $y_i = x_i$ otherwise, we may assume that, for $i \in I$, c_i is a noncut above α_i.

Making F and c n-dimensional

Claim 6.3.4. *It suffices to prove Proposition 6.3.2 in the case that F is non-constant in each coordinate in a neighborhood of c.*

Proof. Suppose that F is constant in the ith coordinate in a neighborhood of c. Then we may take D to be a definable set containing c on which F is continuous and constant in the ith coordinate. Since no c_i is a noncut near $\pm\infty$, we may assume that D is bounded. Let $\pi(x_1, \ldots, x_n) = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\rangle$. Then let $D' = \pi(D)$. Note that $\pi(c) \in D'$. For $d \in D'$, let $\delta(d)$ denote an arbitrary element of $\pi^{-1}(d)$. We can take δ to be definable. Then define

$$F'(d) = F(\delta(d)),$$

which is well-defined by our assumption that F is constant on the ith coordinate in D, and thus on $\pi^{-1}(d)$. Then, by induction, we may find a subset of D' on which F' is continuous, and such that F' extends continuously to the closure of D'. We may take D' to be a cell and replace D by $\pi^{-1}(D') \cap D$. Then by Lemma 1.3.17, $\pi(\overline{D}) = \overline{\pi(D)} = \overline{D'}$. We now show directly that F is continuous on $\overline{D}$. Let $x \in \overline{D}$, and $\epsilon > 0$. We can find an open B around $\pi(x)$ such that $|F'(y) - F'(\pi(x))| < \epsilon$, for $y \in B$. Thus, $|F(z) - F(x)| < \epsilon$, for $z \in \pi^{-1}(B)$, but since π is continuous, $\pi^{-1}(B)$ is an open set containing x, and thus we have found an open set containing x such that $|F(z) - F(x)| < \epsilon$ for z in this set, and thus F is continuous on $\overline{D}$. □

Therefore, we may assume that F is non-constant in each coordinate near c.

Let $M = \mathrm{Pr}(c)$. We can partition M^n into definable cells on which F is continuous and monotonic in each coordinate. The closure of (at least) one of these cells must contain p. Let D be a cell of lowest dimension on which F is continuous, monotonic in each coordinate and whose closure contains p.

Claim 6.3.5. *It suffices to prove Proposition 6.3.2 in the case that D is open.*

Proof. We suppose that D is not open and then show Proposition 6.3.2. Using the p_D defined in [vdD98], Chapter 3, 2.7, we can homeomorphically map D to $p_D(D)$, with $p_D(D) \subseteq D^m$, for $m < n$. Note that p_D is still a homeomorphism on $\overline{D}$. Then, by induction, if $F' = F \circ p_D^{-1}$, we can find definable C' such that $p_D(c) \in C'$, F' is continuous on C', and F' extends continuously as $\overline{F'}$ to $\overline{C'}$. Let $C = p_D^{-1}(C')$. Note that $\overline{C} = p_D^{-1}(\overline{C'})$, by [vdD98], Chapter 6, 1.7 again. Let $\overline{F}(x) = \overline{F'} \circ p_D$. Note that, on C, $\overline{F} = F$. Now we show $\overline{F}$ is continuous. Let $E \subseteq M$ be closed. Then $\overline{F}^{-1}(E) \cap \overline{C}$ is closed iff $p_D(\overline{F}^{-1}(E) \cap C)$ is closed. As in the proof of Claim 6.3.3, this set can be written as $p_D(\overline{F}^{-1}(E)) \cap C'$. Continuing to follow Claim 6.3.3, we can write $p_D(\overline{F}^{-1}(E)) \cap C' = \overline{F}'^{-1}(E) \cap C'$, which is closed by continuity of $\overline{F}'$ on C'. Thus, $\overline{F}$ is continuous on $\overline{C}$. □

Inner Induction

Let f_i', g_i' be the definable lower and upper bounding functions in the construction of D as a cell. We now construct new definable bounding functions f_i, g_i, $1 \le i \le n$, starting at $i = n$ and going down to $i = 1$. Let D^i be the cell defined by replacing the boundary functions f_j', g_j' used to define D with f_j, g_j for $j > i$. For any x with $x_{\le i} \in \pi_{\le i}(D)$, let $E_x^i = \{x\} \times D_x^i$.

We have two induction statements at stage i:

(I1) For any $x \in \pi_{\le i}(D)$, $F(x, -)$ is continuous on $\pi_{>i}(E_x^i)$, and extends continuously to $\overline{\pi^i(E_x^i)}$.

(I2) There is a definable $g(x_{\le i})$, a good bound at i, such that for any $a = \langle a_1, \ldots, a_n\rangle$ and $a' = \langle a_1, \ldots, a_i, a_{i+1}', \ldots, a_n'\rangle$ with $a, a' \in D^i$, $|F(a) - F(a')| < g(a_{\le i})$.

We want to construct definable f_i, g_i to satisfy the inductive conditions for stage $i - 1$. We are constructing $\overline{F}$ as we go, by expanding its domain. At stage i, $\overline{F}$ has domain $\{x \mid x_{\le i} \in \pi_{\le i}(D)\}$, defined as $\overline{F}(b) = \lim_{t\to 0^+} F(\gamma(t))$, where $\gamma(t)$ is any definable curve in E_x^i with $\lim_{t\to 0^+} \gamma(t) = b$ – some such γ exists by Lemma 1.3.16. By (I1), the limit does not depend on choice of γ, so $\overline{F}$ is well-defined at stage i. If we satisfy (I1) for $i-1$, the same argument will show that $\overline{F}$ can be defined on domain $\{x \in \overline{D^i} \mid x_{<i} \in \pi_{\leq i-1}(D)\}$.

We know that, for each x with $x_{\leq i} \in \pi_{\leq i}(D^i)$, F is continuous on $\overline{E^i_x}$. By Lemma 1.3.15, for any $x = \langle x_1, \ldots, x_{i-1}\rangle \in \pi_{\leq i-1}(D)$, we can partition $(f'_i(x_{<i}), g'_i(x_{<i}))$ into intervals (and their endpoints), $I_1(x), \ldots, I_{r(x)}(x)$, so that F is continuous on $\{y \in M^n \mid y_{<i} = x \wedge y_i \in I_j xa)\}$, for $1 \leq j \leq r$. Then we can find an open set $U \subseteq M^i$ containing $c_{\leq i}$ such that $r(x)$ is constant on U, and we denote this constant value by r. Let $I_j(c_{\leq i})$ be given by $(h_j(c_{\leq i}), h_{j+1}(c_{\leq i}))$, for some definable h_j, $j = 1, \ldots, r$, with $h_1 = f'_i$, and $h_{r+1} = g'_i$, with h_j M-definable, for $j = 1, \ldots, r$. Then we may further assume that, on U, $I_j(x) = (h_j(x), h_{j+1}(x))$. Replace D^i by $D^i \cap \{x \mid x_{\leq i} \in U\}$, and replace f'_i, g'_i by h_j, h_{j+1}, respectively, for the j such that $h_j(c_{\leq i}) < c_i < h_{j+1}(c_{\leq i})$. Furthermore, we can assume that, for $a \in \pi_{i-1}(D^i)$ $\overline{D^i_a} = \overline{D^i}_a$, by Lemma 1.3.14.

We must now consider three cases – when c_i is a noncut over $Mc_{<i}$, when it is a uniquely realizable cut, and when it is out of scale on M.

Case 1: $\mathrm{tp}(c_i/c_{<i}M)$ **is a noncut**

We may assume that $f'_i \leq \alpha_i$, since this is true at $c_{<i}$, and so we may actually assume that $f'_i = \alpha_i$. We know that, for any $x = \langle x_1, \ldots, x_{i-1}\rangle \in \pi_{\leq i-1}(D)$, F is continuous on the set

$$\{y \in \overline{D^i} \mid y_{<i} = x \wedge f'_i(y_{<i}) < y_i < g'_i(y_{<i})\}.$$

If we then replace g'_i by $(g'_i + f'_i)/2$, we guarantee that, for x as above, F is continuous on the set

$$\{y \in \overline{D^i} \mid y_{<i} = x \wedge f'_i(y_{<i}) < y_i \leq g_i(y_{<i})\},$$

and furthermore, by our use of Lemma 1.3.14 above, this set is equal to

$$\{y \in \overline{E^i_x} \mid f'_i(y_{<i}) < y_i \leq g_i(y_{<i})\}.$$

Thus, it only remains to show that F extends continuously onto the points where $y_i = f'_i(y_{<i}) = \alpha_i(y_{<i})$. But by Lemma 6.2.2, if we are given x as above, we can restrict D^i further so there is only one such point – $\mathrm{icl}(i, x)$ (note that $N(i) = i$). We have $\mathrm{icl}(i, c) \in \overline{E^i_{c_{<i}}}$. Thus, we can find a $c_{<i}$-definable curve, $\gamma(t, c_{<i})$, such that $\gamma(0, c_{<i}) = \mathrm{icl}(i, c)$, and $\gamma(t, c_{<i}) \in E^i_{c_{<i}}$, for $t > 0$. We may then assume that, for any $y \in D^i$, $\gamma(t, y_{<i})$ is a curve in E^i_y with $\gamma(0, y_{<i}) = \mathrm{icl}(i, y)$. Since F is bounded and continuous, $\lim_{t\to 0^+} F(\gamma(t, y_{<i}))$ exists, for each $y \in D^i$. Let $\gamma_1(t, y_1, \ldots, y_{i-1}), \gamma_2(t, y_1, \ldots, y_{i-1})$ be definable curves in E^i_y with limit at $t = 0$ of $\mathrm{icl}(i, y)$. Fix $a = \langle a_1, \ldots, a_{i-1}\rangle \in \pi_{i-1}(D^i)$. Let $r_j = \lim_{t\to 0^+} F(\gamma_j(t, a)$, $j = 1, 2$. Let ϵ be any positive number. By (I2), there exists a definable g, a good bound at i, such that $|F(y) - F(y')| < g(y_{\leq i})$, for $y, y' \in D^i$ with $a = y_{<i} = y'_{<i}$. Since g is a good bound at i, we can choose $s_1, s_2 > 0$ such that, for $t \in (0, s_j)$, $g(\gamma_j(t, a)_{\leq i}) < \epsilon/3$ and $|F(\gamma_j(t, a)) - r_j| < \epsilon/3$, $j = 1, 2$. Let $s = \min(s_1, s_2)$. Note that $\gamma_1(s, a), \gamma_2(s, a) > \alpha_i(a)$. Assume WLOG that $\gamma_1(s, a) \leq \gamma_2(s, a)$. Then, for some s', $0 < s' \leq s$, $\gamma_2(s', a)$ and $\gamma_1(s, a)$ have the same ith coordinate. Thus, $|F(\gamma_2(s', a)) - F(\gamma_1(s, a))| < g(\gamma_1(s, a)_{\leq i}) < \epsilon/3$, and thus $|r_2 - r_1| < \epsilon$. Thus, $r_2 = r_1$, and so F extends continuously to $\mathrm{icl}(i, x)$, satisfying (I1) for $i - 1$.

We must also satisfy condition (I2) for $i-1$. Let g' be the good bound at $i-1$ with $g' \geq g$ guaranteed from Lemma 6.2.6 (we may restrict D^i so that D^i is the appropriate domain for g'). Let γ be the curve from above. Restrict its domain (in all i coordinates, thus possibly further restricting D^i) so that γ is monotonic in the ith coordinate. Then

$$S(x,z) = \sup\{y : |F(\gamma(t, x_{<i}, y) - F(\text{icl}(i, x_{<i})| < z\}$$

is a function that is decreasing in z for every x. Now replace g_i' with $\min(g_i'(x), S(x, m_{i-1}(x)))$. We have then guaranteed that applying F to any point on γ will yield a value differing little from F applied to the i-closure point.

Then, given $y, y' \in D^{i-1}$ with $y_{<i} = y'_{<i}$, we can find t, t' with $\gamma(t, y_{<i})_{\leq i} = y_{\leq i}$, and similarly for t' and y'. Then

$$\begin{aligned}|F(y)-F(y')| &\leq |F(y)-F(\gamma(t,y_{<i}))|+|F(y')-F(\gamma(t',y_{<i}))|+|F(\gamma(t,y_{<i}))-F(\gamma(t',y'_{<i}))| \\ &\leq g(y_{\leq i}) + g(y'_{\leq i}) + |F(\gamma(t, y_{<i})) - F(\text{icl}(i, y_{<i})| + |F(\gamma(t', y_{<i})) - F(\text{icl}(i, y_{<i})| \\ &\leq 2g'(y_{<i}) + 2m_{i-1}(y_{<i}).\end{aligned}$$

Thus, since $2g' + 2m_{i-1}$ is a good bound at $i-1$, we have satisfied (I2) for $i-1$.

Case 2: $\text{tp}(c_i/c_{<i})$ **is uniquely realizable or out of scale on** P

Condition (I1) for $i-1$ is easily satisfied, because we can choose f_i and g_i such that $\langle x_{<i}, f_i(x_{<i})\rangle$ and $\langle x_{<i}, g_i(x_{<i})\rangle$ are in the interior of D^i, for $x_{<i} \in \pi_{\leq i-1}(D)$. Thus, we already knew, by our restriction on f_i' and g_i', that f was continuous on $\overline{D^{i-1}_{x_{<i}}}$.

If $\text{Pr}(c_{<i})$ realizes no noncuts over P, then Condition (I2) is vacuously satisfied, so we may assume that $\text{Pr}(c_{<i})$ does realize a noncut over P. Then, by Lemma 5.2.10, we know that if $\text{tp}(c_i/c_{<i})$ is uniquely realizable, then no $c_{<i}$-definable function takes P to a set that is cofinal or coinitial at c_i in $\text{Pr}(c_{<i})$. This property holds by definition if $\text{tp}(c_i/c_{<i})$ is out of scale on P. This will let us satisfy Condition (I2).

Define $\mu(x) = \sup\{F(y) \mid y_{\leq i} = x_{\leq i}\}$. The function μ will play a similar role to the curve γ that was used in the noncut case. For $x \in D^i$, note that $|\mu(x_{\leq i}) - F(x)| \leq g(x_{\leq i})$, for some g a good bound at i, by (I2) for i. As well, we can find some g', a good bound at $i-1$, such that $g' \geq g$ on D^i, by Lemma 6.2.6. Thus, if we can bound $|\mu(x_{\leq i}) - \mu(x'_{\leq i})|$ by some good bound at $i-1$, where $x_{<i} = x'_{<i}$, we will be done.

WLOG, assume that F is increasing in the ith coordinate. Now, consider $\mu^{-1}_{c_{<i}}$. Let $k = N(i)$ (from Definition 5.1.21). We know that $k > 0$, since if k were 0, then $\text{Pr}(c_{<i})$ would contain no noncuts.

Let $M' = \text{dcl}(c_{<k})$. By Lemma 5.2.19, we know that $\mu^{-1}_{c_{<i}}(M')$ is neither cofinal nor coinitial at c_i. We can thus take definable functions f_i and g_i such that, for $y_i \in [f_i(c_{<i}), g_i(c_{<i})]$, $\mu(c_{<i}, y_i) \notin M'$, and thus, $\text{tp}(\mu(c_{<i}, y_i)/M') = \text{tp}(\mu(c_{<i}, y_i')/M')$, for any $y_i, y_i' \in [f_i(c_{<i}), g_i(c_{<i})]$, since for two elements to have different types over M', there must be an element of M' between them.

Claim 6.3.6. *For b, b' elements in $[f(c_{<i}), g(c_{<i})]$, $\text{tp}(|\mu(c_{<i}, b) - \mu(c_{<i}, b')|/M')$ is a noncut near* 0.

Proof. Suppose not. Then there is some $r \in (0, |\mu(c_{<i}, b) - \mu(c_{<i}, b')|) \cap M'$. Since μ is a bounded function (since F is), it cannot be the case that $\mu(c_{<i}, b)$ is a noncut near $\pm\infty$ over $\emptyset$. Thus, $\mu(c_{<i}, b)$ must be a nonuniquely realizable cut over M': $\mathrm{tp}(\mu(c_{<i}, b)/M') = \mathrm{tp}(\mu(c_{<i}, b')/M')$ and the two differ by more than r, so addition by r witnesses the type being nonuniquely realizable. But, by Theorem 5.2.11, since $\mathrm{tp}(c_k/M')$ is a noncut, and $\mathrm{tp}(c_j/M'c_{<j})$ is a uniquely realizable cut or out of scale, for $k < j \leq i$, $\mathrm{tp}(c_k, \ldots, c_i/M')$ is definable, and hence $M'(c_k, \ldots, c_i)$ realizes no cuts over M', contradiction. □

Thus,

$$\tilde{\mu}(c_{<i}) = \sup\{|\mu(c_{<i}(x_i) - \mu(c_{<i}, x_i')| : x_i, x_i' \in [f_i(c_{<i}), g_i(c_{<i})]\}$$

is a noncut near 0 over M'.

By induction (on n), we know that $\tilde{\mu}$ is continuous on the closure of some definable set C, containing $c_{<i}$. By Lemma 6.2.2, we know that the point $a = \mathrm{icl}(i, c)_{<i}$ is in $\overline{C}$. As well, since any definable set containing $c_{<i}$ has a in its closure, we must have that $\tilde{\mu}$ applied to a is 0: if not, $\tilde{\mu}(a)$ is M'-definable, so $\tilde{\mu}(c) < \tilde{\mu}(a)/2$. Thus for $\epsilon < \tilde{\mu}(a)/2$, $|\tilde{\mu}(c) - \tilde{\mu}(a)| > \epsilon$, and $c_{<i}$ is in every open neighborhood of a in C. Therefore, $\tilde{\mu}$ is not continuous at a, contradiction.

Thus, $\tilde{\mu}$ extends to a as 0, and is thus a good bound at $i-1$, by definition. Since $2\tilde{\mu}(x_{<i}) > |\mu(x_{\leq i}) - \mu(x'_{\leq i})|$ when $x_{<i} = x'_{<i}$, we can satisfy (I2) for $i-1$: given x, x' as in Condition (I2) for $i-1$,

$$|F(x) - F(x')| \leq |F(x) - \mu(x_{\leq i})| + |F(x') - \mu(x'_{\leq i})| + |\mu(x_{\leq i}) - \mu(x'_{\leq i})| < 3g'(x_{<i}) + 2\tilde{\mu}(x_{<i}),$$

and thus are done. □

This concludes the proof of the "if" direction. We now do the "only if" part of Theorem 6.3.1.

Proposition 6.3.7. *Let p be a decreasing n-type over a set A, and $c = \langle c_1, \ldots, c_n \rangle$ a tuple realizing p, such that, for some i, $\mathrm{tp}(c_i/c_{<i}A)$ is a nonuniquely realizable cut, in scale or near scale on A (equivalently, on $Ac_{<N(i)}$). Then there exists a bounded definable function, F, such that, for any definable set containing c, F is not continuous on the closure of the set.*

Proof. As before, we may assume $A = \emptyset$, let $M = \mathrm{Pr}(c)$, and take "definable" to mean "$\emptyset$-definable." Let M' be the prime model, i.e. $M' = \mathrm{dcl}(\emptyset)$. We will construct a definable i-ary function, extending it to be constant on the last $n-1$ coordinates, so we may assume that $i = n$. Let $k = N(n)$. Here, we may have that $k = 0$. By assumption, there is some $c_{<n}$-definable function, $f_{c_{<n}}$, such that $f_{c_{<n}}(M')$ is cofinal or coinitial at c_n in $\mathrm{dcl}(Mc_{<n})$. WLOG, assume it is coinitial. Define $F(x_1, \ldots, x_n) = f^{-1}_{x_{<n}}(x_n)$. Suppose that C is a definable set containing c. Using Lemma 6.2.2, we replace C by a definable set such that $\overline{C}$ contains exactly one point with first k coordinates $\langle c_{<k}, \alpha(c_{<k}) \rangle$, where α is the definable function near which c_k is a noncut. We may further assume that C is a cell. Let g_n be the function bounding the nth coordinate of C from above. Since $f_{c_{<n}}(M')$ is coinitial at c_n in $\mathrm{dcl}(c_{<n})$, there is some element, r, of M', such that $c_n < f_{c_{<n}}(r) < g_n(c_{<n})$. Since

c_n is a nonuniquely realizable cut over $c_{<n}$, we can find some $c_{<n}$-definable ρ such that $\mathrm{tp}(c_n + \rho/c_{<n}) = \mathrm{tp}(c_n/c_{<n})$. WLOG, assume that $f_{c_{<n}}$ is increasing. Thus, considering $f_{c_{<n}}(r) - \rho$, we see that for some $r' < r$, $c_n < f_{c_{<n}}(r') < f_{c_{<n}}(r) < g_n(c_{<n})$. Note that, since $F(c_{<n}, f_{c_{<n}}(r)) = r$, and $F(c_{<n}, f_{c_{<n}}(r')) = r'$, with $r, r' \in M'$, we must have M'-definable sets $D_1 = \{x \in D^i \mid F(x) = r\}$ and $D_2 = \{x \in D^i \mid F(x) = r'\}$.

Again by Lemma 6.2.2, we may possibly shrink D_1 and D_2, keeping $c_{<k} \in \pi_{<k}(D_1), \pi_{<k}(D_2)$, and then assume that for each set $\overline{D_1}$ and $\overline{D_2}$, there is a unique point in the set with first k coordinates $\langle c_{<k}, \alpha_k(c_{<k})\rangle$. But since both $\overline{D_1}$ and $\overline{D_2}$ are subsets of $\overline{C}$, and $\overline{C}$ has a unique such point, there is a common point in $\overline{D}_1$ and $\overline{D}_2$. Since $F = r$ on D_1, and $F = r'$ on D_2, F cannot be extended continuously to this common point. □

6.4 Application to Curves

We can derive a corollary to Theorem 6.3.1 about curves, but we must first introduce some definitions.

Definition 6.4.1. Let f and g be unary functions, (not necessarily definable), each of whose domains includes some positive neighborhood of 0. We say that f and g are *comparable* if, for some $s > 0$, either for all $t \in (0, s)$, $f(t) < g(t)$; or for all $t \in (0, s)$, $f(t) = g(t)$; or for all $t \in (0, s)$, $f(t) > g(t)$.

Definition 6.4.2. Let M be any o-minimal structure expanding a real closed field, and let $\gamma = \langle \gamma_1, \ldots, \gamma_n\rangle$ be a (not necessarily definable) curve in M^n. Say that γ is *ordered* if, for $i = 2, \ldots, n$, γ_i is comparable to every function in the set

$$\{f(\gamma_{i_1}(t), \ldots, \gamma_{i_k}(t)) \mid f \text{ is an } M\text{-definable } k\text{-ary function}, i_1, \ldots, i_k < i\},$$

and γ_1 is comparable to every M-definable function of t.

Note that whether or not γ is ordered does not depend on the ordering of the coordinates of γ.

Definition 6.4.3. Let $\gamma = \langle \gamma_1, \ldots, \gamma_k\rangle$ be an ordered curve in M^k. Let $\gamma(t)$ denote the sequence $\langle \gamma_1(t), \ldots, \gamma_k(t)\rangle \in M^k$, for $t \in M$. Let

$$\mathrm{tp}(\gamma/M) = \lim_{t\to 0^+} \mathrm{tp}(\gamma(t)/M) = \{\varphi(x) \mid \exists s \forall t \in (0, s)\varphi(\gamma(t))\}.$$

Lemma 6.4.4. $\mathrm{tp}(\gamma/M)$ *is a complete, consistent type.*

Proof. It is clear that it is consistent, so it remains to show completeness. Consider any formula, $\varphi(x_1, \ldots, x_k)$. By cell decomposition, φ is equivalent to a disjunction of cell definitions, say $\bigvee_{i=1}^m C_i$. We may assume by induction on k that $\exists x_k \varphi(x_1, \ldots, x_k)$ is determined by $\mathrm{tp}(\gamma/M)$. If it is not in $\mathrm{tp}(\gamma/M)$, then clearly φ is not either, so we may assume that it is. Since $\exists x_k \varphi(x_1, \ldots, x_k)$ defines the set $\bigvee_{i=1}^m \pi_{<k}(C_i)$, we must have that $\langle \gamma_1(t), \ldots, \gamma_{k-1}(t)\rangle$ lies in the projection of the cells $\pi_{<k}(C_{i_1}), \ldots \pi_{<k}(C_{i_r})$, for $t \in (0, s)$, some positive number s, and $i_1, \ldots, i_r \leq m$. Let the kth coordinate cell definition of C_{i_j} be given by (f^{i_j}, g^{i_j}). Thus, if $f^{i_j}(\gamma_{<k})$ and $g^{i_j}(\gamma_{<k})$ are comparable to γ_k for $j \leq r$, then we are done. But γ is ordered, which is sufficient. □

With this lemma, we can then talk about the type of γ_i over $\gamma_{<i}M$ as well. We are now ready to prove a theorem for curves as a corollary of Theorem 6.3.1.

Theorem 6.4.5. *Let M be an o-minimal structure, and $\gamma(t) = \langle \gamma_1(t), \ldots, \gamma_n(t) \rangle$ a (not necessarily definable) ordered curve in M^n, with $\gamma_i(0) \in M$, $i = 1, \ldots, n$. Then the following two statements are equivalent:*

1. *γ can be reordered so that $\mathrm{tp}(\gamma/M)$ is decreasing, and $\mathrm{tp}(\gamma_i/M\gamma_{<i})$ is a noncut, uniquely realizable cut, or out-of-scale nonuniquely realizable cut over $\mathrm{dcl}(M)$.*

2. *For any bounded M-definable function, F, there is an M-definable subset of M^n, C, such that F is continuous on C, F extends continuously to $\overline{C}$, and $\gamma([0, s)) \subseteq \overline{C}$, for some $s > 0$.*

Proof. For the forward direction, let $p = \mathrm{tp}(\gamma/M)$, which is well-defined by Lemma 6.4.4. Then, since (1) holds, p satisfies the conditions of Theorem 6.3.1, with $A = M$, and so we can find the open set guaranteed by Theorem 6.3.1, which we can assume by cell decomposition to be a cell, defined, say, by functions f_i and g_i, for $1 \leq i \leq n$. This cell will satisfy our requirements if $f_i(\gamma_{<i}(t)) < \gamma_i(t) < g_i(\gamma_{<i}(t))$, for $i \leq n$ and sufficiently small t. But since p must imply $f_i(x_{<i}) < x_i < g(x_{<i})$, we must have it for sufficiently small t, and thus, $\overline{C}$ will contain an initial segment of the curve γ.

Inversely, if (1) does not hold, then reorder γ so that $\mathrm{tp}(\gamma/M)$ is decreasing. Since (1) fails, Theorem 3.1 gives us an F that is not continuous on the closure of any definable set containing p. Since any definable set containing γ in a neighborhood of the origin must contain p, we are done. $\square$

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